Scholars Journal of Physics, Mathematics and Statistics ISSN 2393-8056 (Print)
Abbreviated Key Title: Sch. J. Phys. Math. Stat. ISSN 2393-8064 (Online)

Vertex Arboricity of Planar Graphs without 5-Cycles Intersecting With 6-Cycles

Hongling Chen*, Wenshun Teng, Huijuan Wang, Hongwei Gao
School of Mathematics and Statistics, Qingdao University, Shandong 266071, Qingdao,P. R. China

***Corresponding author**
Hongling Chen

Article History
Received: 16.10.2018
Accepted: 25.10.2018
Published: 30.10.2018

DOI:
10.21276/sjpms.2018.5.5.7

Abstract: The vertex arboricity of graph, denoted by $va(G)$, is the minimum number of forest required to partition the vertex set, which is an improper edge coloring. In this paper, we mainly studied vertex arboricity on planar graphs and we have proved if there is without 5-cycles intersecting with 6-cycles, then $va(G) \le 2$.

Keywords: Planar graph, vertex arboricity, intersecting, cycles.

INTRODUCTION

In this paper, we consider finite, simple, and undirected planar graph. For a real number x, $\lfloor x \rfloor$ is the most integer not more than x. Let G be a graph. We use $\Delta(G)$, $\delta(G)$, $V(G)$, $E(G)$ and $F(G)$, to denote the maximum degree, the minimum degree, the vertex set, the edge set and the face set of G, respectively. Let $v \in V(G)$, then the degree $d(v)$ of v denotes the number of edges associated with the face, and for the degree $d(f)$ of the face of the graph, it denotes the number of edges surrounded by the boundary of the face.

Chinese Library Classification: O157.5
MR（2000S）subject Classification: 05C15

The forest k-coloring of a graph is a mapping ϕ from the vertex set $V(G)$ to the set $\{1, 2, \cdots, k\}$ such that each color class induces an acyclic subgraph, i.e.,a forest. The vertex arboricity $va(G)$ of G is the smallest integer k such that G has a k-coloring, which is defined by Chartrand *et al*. [1] in 1968.

It well known that determining the vertex arboricity of a graph is NP-hard [2]. In 1968, Chartrand *et al*. [1] also proved that for any graph G, the $va(G) \le \lceil (1+\Delta(G))/2 \rceil$, and for any planar graph, the $va(G) \le 3$. For planar graph without 3- cycles, without 5- -cycles, without 6 -cycles, the $va(G) \le 2$, which has been demonstrated in [3, 4]. Besides that for planar graph without 4 -cycles[5] , without 7 -cycles[6], without intersecting 3 -cycles[7], without intersecting 5 -cycles[8], without chordal 6 -cycles[9], the $va(G) \le 2$ has been proved.

Theorem 1 If G is a planar graph without 5 -cycles intersecting with 6 -cycles , then $va(G) \le 2$.

To facilitate the proof below, we give some simple definitions and symbols. If a vertex is of degree k, at least k, and at most k, then we call it k-vertex, k^- – vertex and k^+ – vertex, respectively. Similarly, we define k-face, k^- – face and k^+ -face. If the boundary of face f is $(v_1, v_2, \cdots, v_k)$, then face f can be expressed as $(d(v_1), d(v_2), \cdots, d(v_k))$-face. We use $n_k(v)$ to denote the number of k – vertex adjacent to v; $n_k(f)$ to denote the number of k – vertex incident with f; and $f_k(v)$ to denote the number of k – face incident with f. We say that two circles are intersecting if they share at least one common vertex; two circles are adjacent if they share at least one common edge. and a $k-$cycle be called a chordal $k-$cycle if the $k-$cycle having a chord.

PROOF OF THEOREM

Let G be a smallest counterexample to Theorem 1 with the fewest number of vertex and edges. Clearly, G is $2-$connected, so the boundary of each face of G forms a cycle. We first give the structural properties of G, then use Euler's formula and the discharging rules to gain a contradiction.

Lemma 1 ([6]) $\delta(G) \geq 4$.

Lemma 2 ([6,9]) Suppose that (v_1, v_2, v_3) and (v_1, v_3, v_4) are two adjacent $3-$face having a common edge v_1v_3. If $d(v_1)+d(v_3) \leq 9$, then $d(v_2)+d(v_4) \geq 9$.

Lemma 3 ([10]) G does not contain a $k-$cycles $(v_1, v_2, \cdots, v_k)$ adjacent to a $3-$cycle (v_1, v_2, u) such that $d(u)=4$ and $d(v_i)=4$ for every $i \in \{1, 2, \cdots, k\}$.

In order to complete the proof, we need to make use of discharging method. In the following, we assume that G is a planar graph. Firstly, we give each vertex v a charge $w(v)=2d(v)-6$ and each face f a charge $w(f)=d(f)-6$. Thus by the Euler's formula $|V|-|E|+|F|=2$ and $\sum_{v \in V} d(v) = \sum_{f \in F} d(f) = 2|E(G)|$, we have $\sum_{v \in V}(2d(v)-6) + \sum_{f \in F}(d(f)-6) = -12 < 0$. So $\sum_{x \in V \cup F} ch(x) = -12 < 0$. In the following, we will reassign a new charge to each $x \in V \cup F$ denoted by $ch'(x)$ according to the discharging rules. Because our discharging rules only move change around and do not influence the sum, we have $\sum_{x \in V \cup F} ch(x) = \sum_{x \in V \cup F} ch'(x) = -12 < 0$. If. we will show that $ch'(x) \geq 0$ for each $x \in V \cup F$, then we will gain an obvious contradiction to $0 \leq \sum_{x \in V \cup F} ch'(x) = \sum_{x \in V \cup F} ch(x) = -12 < 0$.

Our discharging rules are as follows

R1. Let v be a $4-$vertex incident with a face f.

(1) $\tau(v \to f) = 0$, if f is a $5-$face and $f_3(v)=1$ and $f_4(v)=2$.

(2) $\tau(v \to f) = \frac{1}{2}$, if f is a $5-$face and $f_4(v) \leq 1$.

(3) $\tau(v \to f) = \frac{1}{2}$, if f is a $4-$face.

(4) $\tau(v \to f) = \frac{1}{2}$, if f is a $3-$face and $f_3(v)=4$.

(5) $\tau(v \to f) = \frac{2}{3}$, if f is a $3-$face and $f_3(v)=3$.

(6) $\tau(v \to f) = 1$, if f is a $3-$face and $f_3(v)=2$.

(7) $\tau(v \to f) = 1$, if f is a $3-$face and $f_3(v)=1$.

R2. Let v be a 5- vertex incident with a face f.

(1) $\tau(v \to f) = \frac{1}{2}$, if f is a $5-$face.

(2) $\tau(v \to f) = \frac{1}{2}$, if f is a $4-$face.

(3) $\tau(v \to f) = \frac{4}{3}$, if f is a $3-$ face and $f_3(v) = 3$.

(4) $\tau(v \to f) = \frac{3}{2}$, if f is a $3-$ face and $f_3(v) = 2$.

(5) $\tau(v \to f) = 2$, if f is a $3-$ face and $f_3(v) = 1$.

R3. Let v be a 6^+- vertex incident with a face f.

(1) $\tau(v \to f) = \frac{1}{2}$, if f is a $5-$ face.

(2) $\tau(v \to f) = \frac{1}{2}$, if f is a $4-$ face.

(3) $\tau(v \to f) = \frac{3}{2}$, if f is a $3-$ face.

After applying the vertex rules, we say a face is bad if a negatively charged small face. We now give the rules for face f. Let face f be a face with $d(f) \geq 7$, and let f_0, f_1, f_2, $\cdots$, $f_{d(f)} = f_0$ be the face adjacent to f in a clockwise order. The charge of the face f can be obtained by the Euler's formula, so the face f can be given its adjacent face at least $\frac{1}{7}$.

R4. Let f be a 7^+- face incident with a face f_i.

(1) $\tau(f \to f_i) = \frac{1}{7}$, if $d(f_i) = 3$ or $d(f_i) = 5$.

(2) $\tau(f \to f_i) = \frac{1}{7}$, if $d(f_{i+1}) = 4$ or $d(f_{i+1}) \geq 7$.

R5. Assume $f = (u, v, w)$ be a $(5^+, 4, 4)-$ face, if $f_3(w) = 4$ and w incident with a $(4, 4, 4)-$ face, let $f_0 = (4, 4, 4)$, then $\tau(f \to f_0) = \frac{1}{6} + \frac{1}{7}$.

R6. Assume $f = (u, v, w)$ be a $(5^+, 4, 4)-$ face, if $f_3(w) = 3$ and w incident with a $(4, 4, 4)-$ face, let $f_0 = (4, 4, 4)$, then $\tau(f \to f_0) = \frac{1}{7}$.

Now we are explaining that there are $ch'(x) \geq 0$ for all $x \in V \cup F$.

Let f be a face of G. If $d(f) \geq 6$, then $ch'(x) = ch(x) \geq 0$. If $d(f) = 4$, then $ch'(f) \geq -2 + \frac{1}{2} \times 4 = 0$ by R1(3), R2(2), and R3(2).

Case 1 Assume $f = (v_1, v_2, v_3, v_4, v_5)$ be a $5-$ face of G.

Case 1.1 $n_4(f) = 5$.

If the five $4-$ vertices incident with f are adjacent with $3-$ face, then $ch'(f) \geq -1 + \frac{1}{2} \times 5$

$+\frac{1}{7}\times 5>0$ by R4(1) and R1(2). If exactly one $4-$ vertices in the incident vertices of f is not adjacent with $3-$ face, then $ch'(f)\geq -1+\frac{1}{2}\times 5+\frac{1}{7}\times 3>0$ by R1(2) and R4(1). If at least two $4-$ vertices in the incident vertex of f are not adjacent with $3-$ face, then $ch'(f)\geq -1+\min$ $\left\{\frac{1}{2}\times 5+\frac{1}{7}\times 2,\frac{1}{2}\times 4+\frac{1}{7},\frac{1}{2}\times 3,\frac{1}{2}\times 5+\frac{1}{7},\frac{1}{2}\times 4,\frac{1}{2}\times 5\right\}>0$ by R1(1), R1(2) , R2(1) and R4(1).

Case 1.2 $n_4(f)=4$.

If the four $4-$ vertices incident with f are adjacent with $3-$ face, then $ch'(f)\geq -1+\frac{1}{2}\times 5+\frac{1}{7}\times 3>0$ by R1(2), R2(1) and R4(1). If exactly one $4-$ vertex in the incident vertices of f is not adjacent with $3-$ face, then $ch'(f)\geq -1+\min\left\{\frac{1}{2}\times 5+\frac{1}{7}\times 2,\frac{1}{2}\times 5+\frac{1}{7}\right\}>0$ by R1(2), R2(1) and R4(1). If at least two $4-$ vertices in the incident vertices of f are not adjacent with $3-$ face, then $ch'(f)\geq -1+\min\left\{\frac{1}{2}\times 3,\frac{1}{2}\times 5+\frac{1}{7},\frac{1}{2}\times 4,\frac{1}{2}\times 5\right\}>0$ by R1(1), R1(2) , R2(1) and R4(1).

Case 1.3 $n_4(f)\leq 3$.

Since face f is incident with at least two $5^{+}-$ vertices, then $ch'(f)\geq -1+\min\left\{\frac{1}{2}\times 5+\frac{1}{7}\times 2,\frac{1}{2}\times 5+\frac{1}{7},\frac{1}{2}\times 3,\frac{1}{2}\times 4,\frac{1}{2}\times 4+\frac{1}{7},\frac{1}{2}\times 5\right\}=0$ by R1(1), R1(2), R2(1) and R2(1).

Case 2 Let $f=(v_1,v_2,v_3)$ be a $3-$ face of G.

Case 2.1 $n_4(f)=3$.

If a $4-$ vertices is incident with four $3-$ faces, then the remaining two $4-$ vertices are incident with at most two $3-$ faces. so $ch'(f)\geq -3+1\times 2+\frac{1}{2}+\frac{1}{7}\times 2+\frac{1}{6}+\frac{1}{7}>0$ by Lemma2, R1(4), R1(5), R1(6), R4 and R5. If a $4-$ vertices is incident with three $3-$ faces, then the remaining two $4-$ vertices are incident with at most two $3-$ faces. So $ch'(f)\geq -3+\min\left\{1\times 2+\frac{2}{3}+\frac{1}{7}\times 3, 1\times 2+\frac{2}{3}+\frac{1}{7}\times 4\right\}>0$ by Lemma2, Lemma 3, R1(4), R1(5), R1(6), R4 and R6. If these three $4-$ vertices are exactly incident with two $3-$ faces, then $ch'(f)\geq -3+1\times 3+\frac{1}{7}\times 3>0$ by R1(6) and R4.

Case 2.2 $n_4(f)=2$.

If a $4-$vertices is incident with four $3-$faces, then the remaining $4-$vertices is incident with at most two $3-$faces. so $ch'(f)\geq -3+1+\frac{4}{3}+\frac{1}{2}+\frac{1}{7}\times 2>0$ by R1(4), R1(6) and R4 . If a $4-$vertices is incident with three $3-$faces, then the remaining $4-$vertices is incident with at most two $3-$faces. so $ch'(f)\geq -3+\min\left\{1+\frac{4}{3}+\frac{2}{3}+\frac{1}{7}\times 2, 1+\frac{4}{3}+\frac{2}{3}+\frac{1}{7}\times 3\right\}>0$ by R1(4), R1(5) and R4 . If these two $4-$vertices are exactly incident with two $3-$faces, then $ch'(f)\geq -3+\frac{4}{3}+$

$1\times 2+\frac{1}{7}>0$ by R1(6), R2(3) ,R3(3) and R4.

Case 2.3 $n_4(f)\leq 1$.

If the $4-$vertices is incident with four $3-$faces, then $ch'(f)\geq -3+\frac{4}{3}\times 2+\frac{1}{2}+\frac{1}{7}>0$ by R1(4), R2(3), R3(3) and R4 .If the $4-$vertices is incident with three $3-$faces, then $ch'(f)\geq -3+\min\left\{\frac{4}{3}\times 2+\frac{2}{3}+\frac{1}{7}, \frac{4}{3}\times 2+\frac{2}{3}+\frac{1}{7}\times 2\right\}>0$ by R1(5), R2(3), R3(3) and R4 .If the $4-$vertices is incident with at most two $3-$-faces, then $ch'(f)\geq -3+\frac{4}{3}\times 2+1>0$ by R1(6), R1(7), R2(3) and R3(3). If there is no $4-$vertex, then $ch'(f)\geq -3+\frac{4}{3}\times 3>0$ by R2(3) and R3(3).

Let v be a vertex of G . Assume $d(v)=4$, if $f_3(v)=4$, then $ch'(v)=2-\frac{1}{2}\times 4=0$

by R1(4). If $f_3(v)=3$, then $ch'(v)=2-\frac{2}{3}\times 3=0$ by R1(5). If $f_3(v)=2$, then $ch'(v)=2-1\times 2=0$ by R1(6). If $f_3(v)=1$, then $ch'(v)\geq 2-1-\frac{1}{2}\times 2=0$ by R1(1), R1(4) and R1(6). If $f_3(v)=0$, then $ch'(v)\geq 2-\frac{1}{2}\times 4=0$ by R1(2) and R1(4). Assume $d(v)=5$, if $f_3(v)=3$, then $ch'(v)=4-\frac{4}{3}\times 3=0$ by R2(3). If $f_3(v)=2$, then $ch'(v)\geq$

$4-\max\left\{\frac{3}{2}\times 2+\frac{1}{2}, \frac{3}{2}\times 2\right\}>0$ by R2(2) and R2(4). If $f_3(v)=1$, then $ch'(v)=4-2-\frac{1}{2}\times 4>0$ by R2(1), R2(2) and R2(5). If $f_3(v)=0$, then $ch'(v)=4-\frac{1}{2}\times 5>0$ by R2(1). Assume $d(v)=6$, if $f_3(v)=4$, then $ch'(v)=6-\frac{3}{2}\times 4>0$ by R3(3). If $f_3(v)=3$, then $ch'(v)=6-\max\left\{\frac{3}{2}\times 3+\frac{1}{2}, \frac{3}{2}\times 3\right\}>0$ by R3(1) and R3(3). If $f_3(v)=2$, then $ch'(v)=6$

$\max\left\{\frac{3}{2}\times 2+\frac{1}{2}\times 2, \frac{3}{2}\times 2+\frac{1}{2}\times 3\right\}>0$ by R3(1) and R3(3). If $f_3(v)=1$, then $ch'(v)=6-$

$\frac{3}{2}+\frac{1}{2}\times 5>0$ by R3(1), R3(2) and R3(3). Assume $d(v)\geq 7$, since G contains no $5-$cycles intersecting with $6-$cycles, if $f_3(v)\leq\left\lfloor\frac{3}{4}d(v)\right\rfloor$, then $f_4(v)=f_5(v)=0$, and we have $2d(v)-6-\frac{3}{2}\times\left\lfloor\frac{3}{4}d(v)\right\rfloor$ $\geq 2d(v)-6-\frac{3}{2}\times\frac{3}{4}d(v)=\frac{7}{8}d(v)-6>0$, so $ch'(v)\geq 0$.

REFERENCES

1. Chartrand G, Kronk HV, Wall CE. The point-arboricity of a graph. Israel Journal of Mathematics. 1968 Apr 1;6(2):169-75.
2. Garey MR. Computers and Intractability: A Guide to the Theory of NP-completeness, Freeman. Fundamental. 1997.
3. Wang W, Lih KW. Choosability and edge choosability of planar graphs without five cycles. Applied mathematics letters. 2002 Jul 1;15(5):561-5.
4. Fijavž G, Juvan M, Mohar B, Škrekovski R. Planar graphs without cycles of specific lengths. European Journal of Combinatorics. 2002 May 1;23(4):377-88.
5. Raspaud A, Wang W. On the vertex-arboricity of planar graphs. European Journal of Combinatorics. 2008 May 1;29(4):1064-75.
6. Huang D, Shiu WC, Wang W. On the vertex-arboricity of planar graphs without 7-cycles. Discrete Mathematics. 2012 Aug 6;312(15):2304-15.
7. Chen M, Raspaud A, Wang W. Vertex-arboricity of planar graphs without intersecting triangles. European Journal of Combinatorics. 2012 Jul 1;33(5):905-23.
8. Cai H, Wu J, Sun L. Vertex arboricity of planar graphs without intersecting 5-cycles. Journal of Combinatorial Optimization. 2018 Feb 1;35(2):365-72.
9. Huang D, Wang W. Vertex arboricity of planar graphs without chordal 6-cycles. International Journal of Computer Mathematics. 2013 Feb 1;90(2):258-72.
10. Choi I, Zhang H. Vertex arboricity of toroidal graphs with a forbidden cycle. Discrete Mathematics. 2014 Oct 28;333:101-5.