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Fixed Point Theorems for Geraghty-Khan Contractions in B-metric Spaces

Liu Jingshuai, Ji Peisheng*
School of Mathematical Sciences, Qingdao University, Shangdong 266071, P. R. China

***Corresponding author**
Ji Peisheng

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Abstract: In this paper, we introduce the concept of Geraghty-Khan contraction in b-metric spaces. By using the method of the fixed point theory, we prove some fixed point results for this contraction in complete b-metric space.
Keywords: Fixed point, Geraghty-Khan contraction, b-metric space.

INTRODUCTION

The Banach contraction principle introduced by Banach [1] is one of the most important results in mathematical analysis. Indeed, it is widely used as the source of metric fixed point theory. In 1973, Geraghty [2] generalized Banach contraction principle as follow:

Theorem1.1[2] Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y),$$

where $\beta : [0, \infty) \to [0,1)$ is a function satisfying $\beta(t_n) \to 1$ implies $t_n \to 0$ as $n \to \infty$. Then T has a unique fixed point. Afterward, some authors have obtained fixed point theorems for some kinds of Geraghty type contractive mappings (see [3,4,10,11,12]).

Recently, H. Piri *et al*. [5] obtained Khan type fixed point theorems in a generalized metric space. Hossein Piri *et al*. [7] introduced the concept of F-Khan-contractions and proved a fixed point theorem in complete metric spaces. Besides, some authors have proved some Khan type fixed point theorems see [6,8].

In 1993, Czerwik introduced in [9] the concept of a b-metric space. Since then, several papers dealt with fixed point theory for single-valued in b-metric spaces see [10-14].

In this paper, we will introduce the Geraghty-Khan contraction and we shall prove the existence of new fixed point theorem for this contraction in complete b-metric space. Through this paper, we denote by N the set of positive integers.

Preliminaries
For the sake of completeness, we recall some basic definitions and lemma.

Definition 2.1. [9] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is a b-metric on X, if for all $x, y, z \in X$, the following conditions hold:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ (b-triangular inequality).

Then (X, d) is called a b-metric space.

Definition 2.2. [12] Let X be a b-metric space and $\{x_n\}$ is a sequence in X. Then

(a) the sequence $\{x_n\}$ is convergent if and only if there exists $z \in X$ such that $\lim_{n\to\infty} d(x_n, z) = 0$;

(b) the sequence $\{x_n\}$ is Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$;

(c) $\{x_n\}$ is complete, if every Cauchy sequence in X is convergent.

The following lemma will be used for proving our main results.

Lemma 1. ([14]) Let (X, d) be a b-metric space with $s \geq 1$, and suppose that sequences $\{x_n\}$ and $\{y_n\}$ are convergent to x, y, respectively. Then we have

$$\frac{1}{s^2} d(x, y) \leq \lim_{n\to\infty} \inf\ d(x_n, y_n) \leq \lim_{n\to\infty} \sup\ d(x_n, y_n) \leq s^2 d(x, y),$$

in particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x, z) \leq \lim_{n\to\infty} \inf\ d(x_n, z) \leq \lim_{n\to\infty} \sup\ d(x_n, z) \leq sd(x, z).$$

MAIN RESULTS

As in [11], for a real number $s > 1$, let F_s denote the class of all functions $\beta : [0, \infty) \to [0, 1/s)$ satisfying the following condition:

$$\beta(t_n) \to \frac{1}{s} \text{ implies } t_n \to 0, \text{ as } n \to \infty.$$

Definition 3.1. Let (X, d) be a b-metric space with $s > 1$, and let $T : X \to X$ be a self-mapping. A mapping T is said to be a Geraghty-Khan contraction, if there exist $\beta \in F_s$ such that for all distinct $x, y \in X$, the following conditions holds:

$$d(Tx, Ty) \leq \begin{cases} \beta(M(x, y)) M(x, y), & \text{if } \max\{d(x, Ty), d(Tx, y)\} \neq 0, \\ 0, & \text{if } \max\{d(x, Ty), d(Tx, y)\} = 0, \end{cases} \tag{3.1}$$

where $M(x, y) = \max\{d(x, y), \dfrac{d(x, Tx) d(x, Ty) + d(y, Ty) d(y, Tx)}{\max\{d(x, Ty), d(Tx, y)\}}\}$.

Theorem 3.1. Let (X, d) be a complete b-metric space with parameter $s > 1$ and $T : X \to X$ be a Geraghty-Khan contraction. Then T has a unique fixed point x^* and for all $x_0 \in X$, the sequence $\{Tx_0\}$ is converges to x^*.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be a sequence in X with $x_{n+1} = Tx_n = T^n x_0$, for all $n \in N$. If $x_n = x_{n-1}$ for some $n \in N$, then $x_{n-1} = Tx_{n-1}$, x_{n-1} is a fixed point of T. Therefore, we will assume that $x_n \neq x_{n-1}$ for all $n \in N$. Then $0 < d(x_{n-1}, x_n) \leq M(x_{n-1}, x_n)$ for all $n \in N$.

We shall divide the proof into two cases.

Case1. Assume that $\max\{d(x_{n-1}, Tx_n), d(Tx_{n-1}, x_n)\} \neq 0$ for all $n \in N$. Then, from (3.1) we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \beta(M(x_{n-1}, x_n)) M(x_{n-1}, x_n), \tag{3.2}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, Tx_{n-1}) d(x_{n-1}, Tx_n) + d(x_n, Tx_n) d(x_n, Tx_{n-1})}{\max\{d(x_{n-1}, Tx_n), d(Tx_{n-1}, x_n)\}}\} \\ &= \max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n) d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1}) d(x_n, x_n)}{\max\{d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}}\} \\ &= d(x_{n-1}, x_n). \end{aligned} \tag{3.3}$$

From (2.2) and (2.3) we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \le \beta(d(x_{n-1}, x_n))\, d(x_{n-1}, x_n) < \frac{1}{s} d(x_{n-1}, x_n) < d(x_{n-1}, x_n)\,. \quad (3.4)$$

Therefore, $\{d(x_n, x_{n+1})\}$ is decreasing. Then there exists $r \ge 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. We will prove that $r = 0$. Suppose on contrary that $r > 0$. Then, letting $n \to \infty$ from (3.4) we have

$$r \le \lim_{n\to\infty} \beta(d(x_{n-1}, x_n))\, r < r\,,$$

which is a contradiction. We have $r = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. (3.5)

Next, we will show that $\{x_n\}$ is a Cauchy sequence in X. We will prove by using a contradiction technique. Assume that $\{x_n\}$ is not a Cauchy sequence. Therefore there exists $\varepsilon > 0$ for we can find two sub-sequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$n_i > m_i > i$, $d(x_{m_i}, x_{n_i}) \ge \varepsilon$. This means that $d(x_{m_i}, x_{n_i-1}) < \varepsilon$. (3.6)

Taking the upper limit in (3.6) as $i \to \infty$, we get $\limsup_{i\to\infty} d(x_{m_i}, x_{n_i-1}) \le \varepsilon$. (3.7)

Using b -triangular inequality, we have

$$\varepsilon \le d(x_{m_i}, x_{n_i}) \le s d(x_{m_i}, x_{m_i+1}) + s d(x_{m_i+1}, x_{n_i})\,.$$

Taking the upper limit in the above inequality and using (3.5), we get

$$\frac{\varepsilon}{s} \le \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i})\,. \quad (3.8)$$

In addition, we have $d(x_{m_i}, x_{n_i}) \le s d(x_{m_i}, x_{n_i-1}) + s d(x_{n_i-1}, x_{n_i})\,.$

By taking the upper limit in the above inequality and using (3.5), we get

$$\limsup_{i\to\infty} d(x_{m_i}, x_{n_i}) \le s\varepsilon\,. \quad (3.9)$$

Using b -triangular inequality, we have

$$d(x_{m_i+1}, x_{n_i-1}) \le s d(x_{m_i+1}, x_{m_i}) + s d(x_{m_i}, x_{n_i-1})\,.$$

Taking the upper limit in the above inequality and using (3.5), we get

$\limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i-1}) \le s\varepsilon$. Hence $0 < \varepsilon \le \max\{d(x_{m_i}, Tx_{n_i-1}), d(Tx_{m_i}, x_{n_i-1})\}$.

Now putting $x = x_{m_i}$ and $y = x_{n_i-1}$ in (3.1), then we obtain

$$d(x_{m_i+1}, x_{n_i}) = d(Tx_{m_i}, Tx_{n_i-1}) \le \beta(M(x_{m_i}, x_{n_i-1}))\, M(x_{m_i}, x_{n_i-1})\,, \quad (3.10)$$

where

$$M(x_{m_i}, x_{n_i-1}) = \max\{d(x_{m_i}, x_{n_i-1}), \frac{d(x_{m_i}, Tx_{m_i}) d(x_{m_i}, Tx_{n_i-1}) + d(x_{n_i-1}, Tx_{n_i-1}) d(x_{n_i-1}, Tx_{m_i})}{\max\{d(x_{m_i}, Tx_{n_i-1}), d(Tx_{m_i}, x_{n_i-1})\}}\}$$

$$= \max\{d(x_{m_i}, x_{n_i-1}), \frac{d(x_{m_i}, x_{m_i+1}) d(x_{m_i}, x_{n_i}) + d(x_{n_i-1}, x_{n_i}) d(x_{n_i-1}, x_{m_i+1})}{\max\{d(x_{m_i}, x_{n_i}), d(x_{m_i+1}, x_{n_i-1})\}}\}$$

$$\le \max\{d(x_{m_i}, x_{n_i-1}), d(x_{m_i}, x_{m_i+1}) + d(x_{n_i-1}, x_{n_i})\}\,. \quad (3.11)$$

Taking the upper limit $i \to \infty$ in (3.11), by using (3.5) and (3.7) we get

$$\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}) \le \varepsilon\,. \quad (3.12)$$

Next, taking the upper limit as $i \to \infty$ in (3.10) and using (3.8), (3.11) and (3.12) we have

$$\frac{\varepsilon}{s} \le \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i}) \le \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1}))M(x_{m_i}, x_{n_i-1})$$

$$< \frac{1}{s}\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1})$$

$$\le \frac{\varepsilon}{s} < \varepsilon,$$

which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} d(x_n, x^*) = 0$. (3.13)

We show that x^* is a fixed point of T.

If for each $n \in N$, there exists $i_n \in N$ such that $x_{i_n} = Tx^*$ and $i_n > i_{n-1}$, then we have

$$x^* = \lim_{n\to\infty} x_{i_{n+1}} = Tx^*.$$

This proves that x^* is a fixed point of T.

Suppose now, there exists $\mathrm{N} \in N$ such that $x_n \ne Tx^*$. This implies $d(x_n, Tx^*) > 0, \quad \forall n > \mathrm{N}$. For all $n > \mathrm{N}$, we have $\max\{d(x_n, Tx^*), d(Tx_n, x^*)\} > 0$.

From (3.1), we get

$$d(x_{n+1}, Tx^*) = d(Tx_n, Tx^*) \le \beta(M(x_n, x^*))M(x_n, x^*), \tag{3.14}$$

where

$$M(x_n, x^*) = \max\{d(x_n, x^*), \frac{d(x_n, Tx_n)d(x_n, Tx^*) + d(x^*, Tx^*)d(x^*, Tx_n)}{\max\{d(x_n, Tx^*), d(Tx_n, x^*)\}}\}$$

$$= \max\{d(x_n, x^*), \frac{d(x_n, x_{n+1})d(x_n, Tx^*) + d(x^*, Tx^*)d(x^*, x_{n+1})}{\max\{d(x_n, Tx^*), d(x_{n+1}, x^*)\}}\}.$$

So from (3.5), (3.13) and by lemma 1, taking limits as $n \to \infty$ to each side of (3.14), we get $d(x^*, Tx^*) \le 0$ and hence $d(x^*, Tx^*) = 0$. Now, we show that T has a unique fixed point. For this we assume that y^* is another fixed point of T in X such that $d(x^*, y^*) > 0$. Therefore

$$\max\{d(x^*, Ty^*), d(Tx^*, y^*)\} = d(x^*, y^*) > 0.$$

From (3.1), we have

$$d(x^*, y^*) = d(Tx^*, Ty^*) \le \beta(M(x^*, y^*))M(x^*, y^*), \tag{3.15}$$

where

$$M(x^*, y^*) = \max\{d(x^*, y^*), \frac{d(x^*, Tx^*)d(x^*, Ty^*) + d(y^*, Ty^*)d(y^*, Tx^*)}{\max\{d(x^*, Ty^*), d(Tx^*, y^*)\}}\}$$

$$= d(x^*, y^*). \tag{3.16}$$

From (3.15) and (3.16), we have

$$d(x^*, y^*) = d(Tx^*, Ty^*)$$

$$\le \beta(d(x^*, y^*))d(x^*, y^*)$$

$$< \frac{1}{s}d(x^*, y^*),$$

since $s > 1$, which is a contradiction. Hence $d(x^*, y^*) = 0$. This complete the proof.

Case 2 Assume that there exists $m \in N$ such that

$$\max\{d(x_{m-1}, Tx_m), d(Tx_{m-1}, x_m)\} = 0.$$

By condition (3.1), it follows that $d(Tx_{m-1}, Tx_m) = 0$ and hence $x_m = Tx_m$. This completes the existence of a fixed point of T. The uniqueness follows as in Case 1.

Corollary 3.2 Let (X, d) be a complete b-metric space with parameter $s > 1$ and let $T : X \to X$ be a self-mapping such that

$$d(Tx, Ty) \le \begin{cases} k \max\{ d(x, y), \dfrac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{\max\{ d(x, Ty), d(Tx, y)\}}\}, & \text{if } \max\{ d(x, Ty), d(Tx, y)\} \ne 0, \\ 0, & \text{if } \max\{ d(x, Ty), d(Tx, y)\} = 0, \end{cases}$$

for all $x, y \in X$ and $x \ne y$, and for some $k \in [0, 1/s)$. The T has a unique fixed point x^* and for all $x_0 \in X$, the sequence $\{Tx_0\}$ is converges to x^*.

Proof. It is sufficient to take $\beta(t) = k$ for all $t \in [0, \infty)$ in Theorem 3.1.

Theorem 3.3 Let (X, d) be a complete b-metric space with parameter $s > 1$ and let $T : X \to X$ be a self-mapping such that

$$d(Tx, Ty) \le \begin{cases} \delta d(x, y) + \gamma \dfrac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{\max\{ d(x, Ty), d(Tx, y)\}}, & \text{if } \max\{ d(x, Ty), d(Tx, y)\} \ne 0, \\ 0, & \text{if } \max\{ d(x, Ty), d(Tx, y)\} = 0, \end{cases}$$

for all $x, y \in X$ and $x \ne y$, and for some $\delta, \gamma \in [0, 1/s)$ such that $\delta + \gamma < 1/s$. The T has a unique fixed point x^* and for all $x_0 \in X$, the sequence $\{Tx_0\}$ is converges to x^*.

Proof. Since

$$\delta d(x, y) + \gamma \frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{\max\{ d(x, Ty), d(Tx, y)\}}$$
$$\le (\delta + \gamma) \max\{ d(x, y), \frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{\max\{ d(x, Ty), d(Tx, y)\}}\}.$$

So by taking $k = \delta + \gamma$ in Corollary 3.2, the proof is complete.

Corollary 3.4 ([13]) Let (X, d) be a complete b-metric space with parameter $s > 1$ and let $T : X \to X$ be a self-mapping such that

$$d(Tx, Ty) \le \begin{cases} \delta d(x, y) + \gamma \dfrac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(Tx, y)\}}, & \text{if } d(x, Ty) + d(Tx, y) \ne 0, \\ 0, & \text{if } d(x, Ty) + d(Tx, y) = 0, \end{cases}$$

for all $x, y \in X$ and $x \ne y$, and for some $\delta, \gamma \in [0, 1/s)$ such that $\delta + \gamma < 1/s$. The T has a unique fixed point x^* and for all $x_0 \in X$, the sequence $\{Tx_0\}$ is converges to x^*.

Proof. Since

$$\frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(Tx, y)} \le \frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{\max\{ d(x, Ty), d(Tx, y)\}}.$$

So from Theorem 3.3, the proof is complete.

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