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Bounded traveling wave solutions of the (2+1)-dimensional breaking soliton equation

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Abstract **Review Article**

In this paper, the (2+1)-dimensional breaking soliton equation is studied by the bifurcation theory of dynamical system. Based on this theory, phase portraits of different topological structures of the equation are obtained, which clearly show all bounded orbits corresponding to the bounded traveling waves of the equation. Furthermore, the periodic wave solution of the (2+1)-dimensional breaking soliton equation are obtained by calculating complex elliptic integrals.
Keywords: (2+1)-dimensional breaking soliton equation, dynamical system, traveling wave solutions.

INTRODUCTION

In recent decades, nonlinear evolution equations (NLEE) have been widely used in many scientific fields, such as plasma physics, nonlinear optics, fluid dynamics, solid-state physics, and chaos theory and so on. By studying this kind of nonlinear partial differential equation, we find that the traveling wave solution of partial differential equation plays an important role in the study of the solution's long time behavior and complex nonlinear wave phenomenon. The search for exact travelling wave solutions of partial differential equation has been widely concerned by scholars, and many effective methods to reveal the characteristics and properties of these equations have been obtained, such as Hirota bilinear method [1], tanh function method [2], Jacobi elliptic function expansion method [3], homogeneous balance method [4], the F-expansion method [5,6], Exp function method [7], bifurcation theory of dynamical system [8,9,10,11], etc.

This paper considers the following (2+1)-dimensional breaking soliton equation

$$U_{xxxy} - 4U_xU_{xy} - 2U_{xx}U_y + U_{xt} = 0, \qquad (1.1)$$

This equation was used to describe the (2+1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis [12-18]. For $y = x$, and by integrating the resulting equation in Eq. (1.1), the equation is reduced to the KdV equation. The Painlevé property, the Lax pair, the Hamiltonian structure, and various exact solutions have been studied [16, 19-27]. A class of overturning soliton solutions has been introduced in Refs [16, 28]. Moreover, Eq. (1) was studied in [15] using the homogeneous balance principle followed by the simplified Hirota's method. The analytic interaction solutions between solitons and cnoidal periodic waves for the (2+1)-dimensional breaking soliton equation are shown in [29] by means of the nonlocal symmetry method. In reference [30], a simplified Hirota method is used to obtain multiple soliton solutions for each developed breaking soliton equation. The generalized dispersion relation is established for the typical breaking soliton equations and the generalized negative-order breaking soliton equations.

Although there are many profound results about the traveling wave solutions of e Eq. (1.1), which are helpful for our understanding of nonlinear physical phenomena and wave propagation, it is a pity that the traveling wave solutions of Eq. (1.1) is not fully discussed, especially for its bounded traveling wave solutions. Therefore, the purpose of this paper is to seek all possible bounded traveling wave solutions in Eq. (1.1). Motivated by them, our strategy is to transform the traveling wave equation of Eq. (1.1) into a dynamical system in R^3. Fortunately, there exists a 2-dimensional invariant manifold which determines most of dynamical behaviours. Then, bifurcation analysis is applied to seek the parameter bifurcation sets which determine various qualitatively different phase portraits. Finally, by calculating the complex elliptic integrals along these orbits, the expressions of all bounded traveling wave solutions in the (2+1)-dimensional breaking soliton equation are obtained.

Traveling wave system and bifurcation analysis

With the following traveling wave transformation

$$U = U(t, x, y) = u(\xi) = u(x + ay - ct),$$

equation (1) can be transformed into its raveling wave system

$$au'''' - 4au'u'' - 2au''u' - cu'' = 0, \tag{2.1}$$

where $'$ stands for $d/d\xi$, $a \neq 0$ represent the wave numbers in the direction y respectively and $c \neq 0$ is the wave speed. Integrating (2.1) once and retaining an integral constant, the following equation is obtained

$$au''' - 3a(u')^2 - cu' = e, \tag{2.2}$$

where parameter e is the integral constant, equation (2.2) has the following equivalent form

$$\begin{cases} u' = p, \\ p' = q, \\ q' = 3p^2 + \dfrac{c}{a}p + \dfrac{e}{a}, \end{cases} \tag{2.3}$$

which is a dynamical system in R^3. Obviously, system (2.3) has a 2-dimensional invariant manifold in R^3. Flows on it can be determined by the last two equations in system (2.3), i.e.

$$\begin{cases} p' = q, \\ q' = 3p^2 + \dfrac{c}{a}p + \dfrac{e}{a}, \end{cases} \tag{2.4}$$

which is exactly a Hamiltonian system with the energy function

$$H(p, q) = \frac{1}{2}q^2 - p^3 - \frac{c}{2a}p^2 - \frac{e}{a}p. \tag{2.5}$$

Firstly, we start with equilibrium of system (2.4).

Theorem 2.1. When $c^2 - 12ae > 0$, system (2.4) has two equilibrium, a saddle $B_1\left(-\frac{c}{6a} + \sqrt{\frac{c^2-12ae}{36a^2}}, 0\right)$, and a center $B_2\left(-\frac{c}{6a} - \sqrt{\frac{c^2-12ae}{36a^2}}, 0\right)$. When $c^2 - 12ae = 0$, system (2.4) has a unique equilibrium of higher order $B_3\left(-\frac{c}{6a}, 0\right)$. When $c^2 - 12ae < 0$, system (2.4) has no equilibrium.

Proof. When $c^2 - 12ae > 0$, a direct calculation shows that system (5) has two equilibrium $B_1\left(-\frac{c}{6a} + \sqrt{\frac{c^2-12ae}{36a^2}}, 0\right)$, $B_2\left(-\frac{c}{6a} - \sqrt{\frac{c^2-12ae}{36a^2}}, 0\right)$. Let $M(p, q)$ to denote the Jacabian matrix of system (2.4) at point (p, q), we have

$$M(B_1) = \begin{bmatrix} 0 & 1 \\ \sqrt{\dfrac{c^2 - 12ae}{a^2}} & 0 \end{bmatrix},$$

$$M(B_2) = \begin{bmatrix} 0 & 1 \\ -\sqrt{\dfrac{c^2 - 12ae}{a^2}} & 0 \end{bmatrix}.$$

From this, we can find

$$det\, M\,(B_1) = -\sqrt{\frac{c^2 - 12ae}{a^2}} < 0$$
$$det\, M\,(B_2) = \sqrt{\frac{c^2 - 12ae}{a^2}} > 0.$$

By the theory of plane dynamic system [31-33] and the properties of Hamiltonian system [32], it is not difficult to check that B_1 is a saddle and B_2 is a center in this case.

When $c^2 - 12ae = 0$, the system (2.4) has only one equilibrium $B_3\left(-\frac{c}{6a}, 0\right)$ with a nilpotent matrix

$$M(B_3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

this shows that B_3 is a high-order equilibrium. In order to be able to further determine the type of B_3, we do the following homeomorphic transformation

$$\xi = p + \frac{c}{6a}, \eta = q,$$

at this point, the system (2.4) can be transformed into its normal form below

$$\begin{cases} \xi' = \eta, \\ \eta' = 3\xi^2. \end{cases}$$

By the qualitative theory of differential equation [33 Theorem 7.3, Chapter 2], we have $k = 2$ and $b_n = 0$, which indicates that B_3 is a cusp.

When $c^2 - 12ae < 0$, it is easy to see that there is no equilibrium of system (2.4).

Next we need to illustrate the parameter bifurcation sets with$\{(a, c, e)|c^2 - 12ae > 0\}$, $\{(a, c, e)|c^2 - 12ae = 0\}$ and$\{(a, c, e)|c^2 - 12ae < 0\}$.

Based on the analysis of the equilibrium and the properties of the Hamiltonian system [19], we have the following results.

Case 1: Consider $c^2 - 12ae > 0$, there is a homoclinic orbit γconnected to the saddle B_1. The center B_2 is surrounded by the family of periodic orbits

$$\Gamma(h) = \{H(p, q) = h, h \in \left(h(B_2,), h(B_1,)\right)\},$$

Where

$$h(B_1) = \frac{-c^3 + (ac^2 - 12ea^2)\sqrt{\frac{c^2 - 12ae}{a^2}} + 18aec}{108a^3},$$
$$h(B_2) = \frac{-c^3 - (ac^2 - 12ea^2)\sqrt{\frac{c^2 - 12ae}{a^2}} + 18aec}{108a^3}.$$

Moreover, $\Gamma(h)$ tends to B_2 as $h \to h(B_2)$ and tends to γ as $h \to h(B_1)$, besides the homoclinic orbit and periodic orbits, other orbits of system (2.4) are unbounded, as shown in figure1(a).

Case 2: Consider $c^2 - 12ae = 0$, the system (2.4) has two types of orbits, of which orbit L was different from other orbits, but all the orbits here were unbounded, as show in figure1(b).

Case 3: Consider $c^2 - 12ae < 0$, system (2.4) has only one type of orbits, and each orbit is unbounded, as show in figure1(c).

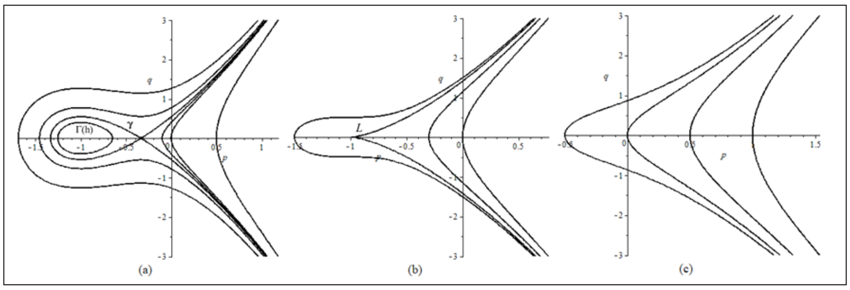

Fig-1: The phase portraits of (2.4)

It can be seen from the phase diagram that only case 1 has bounded orbits, namely a family of periodic orbits $\Gamma(h)$ and a homologous orbit γ {see figure1 (a)}, which correspond to the periodic wave and shock wave of system (2.4) respectively. The expressions of traveling wave solutions corresponding to these bounded orbitals are given below.

Explicit traveling wave solutions of Eq. (1.1)

In this section, we will give the explicit expression of all bounded traveling wave solutions for Eq. (1.1), according to the system (2.4), in order to derive the final traveling wave solutions $u(\xi)$ of the (2+1)-dimensional breaking soliton equation, we need to integrate the solutions of system (2.4) once with respect to ξ.

3.1 Consider the periodic orbits, from the energy function (2.5), any one of the periodic orbits $\Gamma(h)$ can be expressed by

$$q = \pm\sqrt{2(p-p_1)(p_2-p)(p_3-p)},$$

Where p_1, p_2 and p_3 are reals and $p_1 < p < p_2 < p_3$. Assume that the period of these closed orbits is 2T, and choose$p(0) = p_1$, we have

$$\int_{p_1}^{p} \frac{dp}{\sqrt{2(p-p_1)(p_2-p)(p_3-p)}} = \int_0^{\xi} d\xi\,, \qquad 0 < \xi < T.$$

$$-\int_{p}^{p_1} \frac{dp}{\sqrt{2(p-p_1)(p_2-p)(p_3-p)}} = \int_{\xi}^{0} d\xi\,, \qquad -T < \xi < 0.$$

Which can be rewritten as

$$\int_{p_1}^{p} \frac{dp}{\sqrt{2(p-p_1)(p_2-p)(p_3-p)}} = |\xi|, \qquad -T < \xi < T.$$

By calculating the elliptic integral

$$\int_{p_1}^{p} \frac{dp}{\sqrt{(p-p_1)(p_2-p)(p_3-p)}} = g \cdot sn^{-1}\left(\sqrt{\frac{p-p_1}{p_2-p_1}}, k\right),$$

Where $k^2 = \frac{p_2-p_1}{p_3-p_1}$ and $g = \frac{2}{\sqrt{p_3-p_1}}$, we get the expression of periodic wave solution of the system (2.4)

$$p_1(\xi) = p_1 + (p_2-p_1)sn^2\left(\sqrt{\frac{p_3-p_1}{2}}|\xi|\right), \qquad -T < \xi < T. \tag{3.1}$$

It is not difficult to check that expression (3.1) can be further simplified to

$$p_1(\xi) = p_1 + (p_2-p_1)sn^2\left(\sqrt{\frac{p_3-p_1}{2}}\xi\right), \qquad -T < \xi < T.$$

Then, the first type of traveling wave solution of Eq. (1.1) can be calculated by

$$u_1(\xi) = \int p_1(\xi)\, d\xi == \int \left[p_1 + (p_2-p_1)sn^2\left(\sqrt{\frac{p_3-p_1}{2}}\xi\right)\right] d\xi,$$

By calculating the elliptic integral

$$\int sn^2\, udu = \frac{1}{k^2}[u - E(u)]$$

where $E(u) = E(\phi, k)$. We can conclude that

$$u_1(\xi) = p_3 \cdot \xi - \sqrt{2(p_3 - p_1)} E\left(\sqrt{\frac{p_3 - p_1}{2}} \xi\right), \qquad -T < \xi < T.$$

Consider the homologous orbit, by (2.5), the homologous orbit γ can be expressed by

$$q = \pm\sqrt{2(p - p_5)^2(p - p_4)} = \pm\sqrt{2}(p - p_5)\sqrt{p - p_4},$$

Where p_4, p_5 are reals, $-\infty < p_4 < p_5 < p$, $p_4 = -\frac{c}{6a} - 2\sqrt{\frac{c^2 - 12ae}{a^2}}$ and $p_5 = -\frac{c}{6a} + \sqrt{\frac{c^2 - 12ae}{a^2}}$, and choosing initial value $p(0) = p_5$, we have

$$\int_{p_5}^{p} \frac{dp}{\sqrt{2}(p - p_5)\sqrt{p - p_4}} = \int_0^{\xi} d\xi, \qquad \xi > 0,$$

$$-\int_{p}^{p_5} \frac{dp}{\sqrt{2}(p - p_5)\sqrt{p - p_4}} = \int_{\xi}^{0} d\xi, \qquad \xi < 0,$$

Which can be rewritten as

$$\int_{p_5}^{p} \frac{dp}{\sqrt{2}(p - p_5)\sqrt{p - p_4}} = |\xi|, \qquad -\infty < \xi < +\infty.$$

Noting that

$$\int_{p_5}^{p} \frac{dp}{(p - p_5)\sqrt{p - p_4}} = \frac{1}{\sqrt{p_5 - p_4}} \ln \frac{\sqrt{p - p_4} - \sqrt{p_5 - p_4}}{\sqrt{p - p_4} + \sqrt{p_5 - p_4}},$$

we get the expression of solitary wave solution of the system (2.4)

$$p_2(\xi) = p_4 + \frac{(p_5 - p_4)\left(1 + \exp\left(\sqrt{2(p_5 - p_4)}|\xi|\right)\right)^2}{\left(1 - \exp\left(\sqrt{2(p_5 - p_4)}|\xi|\right)\right)^2}, \qquad -\infty < \xi < +\infty.$$

It's easy to check that $p_2(\xi) = p_2(-\xi)$, It means that $p_2(\xi)$ can be simplified to the following form

$$p_2(\xi) = p_4 + \frac{(p_5 - p_4)\left(1 + \exp\left(\sqrt{2(p_5 - p_4)}\xi\right)\right)^2}{\left(1 - \exp\left(\sqrt{2(p_5 - p_4)}\xi\right)\right)^2}, \qquad -\infty < \xi < +\infty.$$

Then, the second type of traveling wave solution of Eq. (1.1) can be calculated by

$$u_2(\xi) = \int p_2(\xi)\, d\xi = \int \left(p_4 + \frac{(p_5 - p_4)\left(1 + \exp\left(\sqrt{2(p_5 - p_4)}\xi\right)\right)^2}{\left(1 - \exp\left(\sqrt{2(p_5 - p_4)}\xi\right)\right)^2}\right) d\xi$$

$$= p_4 \cdot \xi - \sqrt{\frac{p_5 - p_4}{2}} \left[\frac{\sqrt{2(p_5 - p_4)}\xi + \exp\left(\sqrt{2(p_5 - p_4)}\xi\right) + \frac{1}{2}\exp\left(2\sqrt{2(p_5 - p_4)}\xi\right)}{1 - \exp\left(\sqrt{2(p_5 - p_4)}\xi\right)}\right] + C_2$$

where $-\infty < \xi < +\infty$ and C_2 is a constant.

CONCLUSIONS

In this paper, we apply the dynamical system methods to investigate all bounded traveling waves of the (2+1)-dimensional breaking soliton equation. Although it is a high dimensional dynamical system, we find that there existsts a 2-dimensional invanrant manifold which makes it possible to completely investigate all bounded orbits of it by detailed analysing the phase space geometry, and all possible bounded traveling waves of the (2+1)-dimensional breaking soliton equation and corresponding existence conditions can be identified clearly. Among them, using complex elliptic function uniformly is a new solution.

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