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The Number of Limit Cycles for a Class of Quintic Polynomial Systems

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Abstract **Review Article**

This paper mainly uses the classical method of qualitative theory of ordinary differential equations to qualitatively analyze a class of planar fifth-order polynomial systems. The successor function method is used to analyze the system's fine focus. The center of the system is judged by the principle of symmetry. The Hopf branch theory is used to analyze the existence of the limit cycle under different parameters.
Keywords: Five times system, hopf bifurcation, successor function, limit cycle.

INTRODUCTION

In recent decades, with the rapid development of science and technology, the theory of limit cycles has been more widely applied. In the study of qualitative theory of ordinary differential equations, we mainly include two aspects on the study of limit cycles. The first one is the existence, uniqueness, number and relative position of the limit cycle. The second is the limit cycle. The problem of limit ring generation and disappearance as the relevant parameters change [1-5]. The progress around Hilbert's 16th issue was limited before the 1950s. After the 1950s, Soviet mathematicians and Chinese mathematicians made many contributions to the study of limit cycle theory. For the Hilbert question 16 of n = 2, the qualitative study of the planar quadratic system, the mathematicians in China have achieved fruitful results. For the case of n ≥ 3, the better result before 1980 was the work of the Soviet K.C.CHБИРСК. He proved that the cubic system lacking the quadratic term can have five limit cycles in the field where the origin is sufficiently small. In 1975, Shi Songling gave a concrete example of how the above distribution can be achieved. After the 1980s, the representative work was that Li Jibin discovered that the cubic system has a much more complex limit cycle distribution than the quadratic system, and found that (E_3) has a compound eye branch with n limit cycles. In 1989, Li Jibin and Bai Jingxin gave specific examples to illustrate a class of specific cubic polynomial systems [6-10]. In the small small neighborhood of the center point, seven small amplitude limit cycles can be generated by the Unfolding branch [11, 12], etc. At present, the research on the third-order polynomial systems has achieved good results. For fifth-order polynomial systems, only a small number of studies were obtained. The main reason is that singularity calculation is complicated or difficult to implement.

In the study of the differential equation plane system, the determination of the central focus is an important research topic. The order of the focus quantity determines the number of limit cycles generated in the field of singularity under a small disturbance [13-15]. This paper conducts qualitative analysis on the following fifth-order polynomial systems

$$\begin{cases} \dfrac{dx}{dt} = y + xy + x^2y, & (1.1) \\ \dfrac{dy}{dt} = -x + \delta y + a_1x^2y + a_2xy^3 + x^4 + a_3x^4y\,. & (1.2) \end{cases}$$

Where δ and $a_i(i = 1,2,3.)$ are arbitrary real numbers.

Type of system balance point

When $x = y = 0$, Equation (1.1) and (1.2) are equal to 0, then $A(0,0)$ is the equilibrium point of the system. The linear approximation equation matrix of the system is

$$M(A) = \begin{bmatrix} 0 & 1 \\ -1 & \delta \end{bmatrix}$$

The determinant of this matrix is

$$D = 0 \cdot \delta - 1 \times (-1) = 1$$

The trace of this matrix is

$$T_r = 0 + \delta = \delta$$

We make

$$\Delta = {T_r}^2 - 4D = \delta^2 - 4$$

Theorem 1:

(1) When $-2 < \delta < 0$, $A(0,0)$ is a stable coarse focus.
(2) When $0 < \delta < 2$, $A(0,0)$ is an unstable coarse focus.
(3) When $\delta \geqq 2$, $A(0,0)$ is an unstable node.
(4) When $\delta \leq -2$, $A(0,0)$ is a stable node.
(5) When $\delta = 0$, $A(0,0)$ is the center-type equilibrium point.

Proof as follows:

The balance point type of the system is determined by determinant, trace, and discriminant. The symbol of the determinant D is divided into three cases:

Case 1: When $D > 0$,

At this time, the symbols of the trace T_r are divided into three cases:

Case 1.1: When $T_r > 0$, balance point is unable, if $\Delta \geq 0$, blance point is a unable node; if $\Delta < 0$, blance point is a unable coarse focus.

Case 1.2 When $T_r = 0$, balance point is the center-type equilibrium point.

Case 1.3 When $T_r < 0$, balance point is stable, if $\Delta \geq 0$, blance point is a stable node; if $\Delta < 0$, blance point is a stable coarse focus.

Case 2: When $D = 0$,

At this time, blance point is degraded.

Case 3: When $D < 0$,

At this time, blance point is saddle point.

For the balance point A of the system, $D = 1 > 0$.

(1)When $T_r > 0$,which is $\delta > 0$, if $0 < \delta < 2$, then $\Delta < 0$, then A is a unstable coarse focus; if $\delta \geqq 2$, then $\Delta \geq 0$, then A is a unstable node.

(2) When $T_r < 0$,which is $\delta < 0$, if $-2 < \delta < 0$, then $\Delta < 0$, then A is a stable coarse focus; if $\delta \leq -2$, then $\Delta \geq 0$, then A is a stable node.

(3) When $T_r = 0$, which is $\delta = 0$, then A is the center-type equilibrium point.

Center and focus discrimination

When $\delta = 0$, $A(0,0)$ is the center of the linear system corresponding to the system. The following need to determine whether the original system balance point $A(0,0)$ is the center or the focus.This paper uses the successor function to study the properties of the equilibrium point A.

Theorem 2 For this system, when $\delta = 0$, the following conclusions

(1)When $a_1 > 0$, A(0,0) is the first-order unstable fine focus.
(2)When $a_1 < 0$, A(0,0) is the first-order stable fine focus.
(3)When $a_1 = 0, a_3 > 0$, A(0,0) is the second-order unstable fine focus.
(4) When $a_1 = 0, a_3 < 0$, A(0,0) is the second-order stable fine focus.
(5)When $a_1 = a_3 = 0, a_2 > 0$, A(0,0) is the third-order unstable fine focus.
(6) When $a_1 = a_3 = 0, a_2 < 0$, A(0,0) is the third-order stable fine focus.
(7)When $a_1 = a_2 = a_3 = 0$, A(0,0) is the center of the system.

Proof as follows:

When $\delta = 0$, the original system is

$$\begin{cases} \dfrac{dx}{dt} = y + xy + x^2y, & (3.1) \\ \dfrac{dy}{dt} = -x + a_1x^2y + a_2xy^3 + x^4 + a_3x^4y\,. & (3.2) \end{cases}$$

Homeomorphic transformation

Make $x_1 = -x, y_1 = y$, then the expression (3.1) and (3.2) are transformed into the following form

$$\begin{cases} \dfrac{dx_1}{dt} = -y_1 + x_1y_1 - {x_1}^2y_1, & (3.3) \\ \dfrac{dy_1}{dt} = x_1 + a_1{x_1}^2y_1 - a_2x_1{y_1}^3 + {x_1}^4 + a_3{x_1}^4y_1\,. & (3.4) \end{cases}$$

Polar Coordinate Transformation

Letting $x_1 = r \cdot cos\theta, y_1 = r \cdot sin\theta$ in equation (3.3) and (3.4), then

$$\begin{cases} \dfrac{dr}{dt} = r^2cos^2\theta sin\theta + r^3(a_1cos^2\theta sin^2\theta - cos^3\theta sin\theta) + r^4(cos^4\theta sin\theta - a_2cos\theta sin^4\theta) + a_3r^5cos^4\theta sin^2\theta, & (3.5) \\ \dfrac{d\theta}{dt} = 1 - r\,cos\theta sin^2\theta + r^2(a_1cos^3\theta sin\theta + cos^2\theta sin^2\theta) + r^3(cos^5\theta - a_2cos^2\theta sin^3\theta) + a_3r^4cos^5\theta sin\theta. & (3.6) \end{cases}$$

Eliminate t and expand into a power series of r, we get the following formula

$$\frac{dr}{d\theta} = \frac{r^2cos^2\theta sin\theta + r^3(a_1cos^2\theta sin^2\theta - cos^3\theta sin\theta) + r^4(cos^4\theta sin\theta - a_2cos\theta sin^4\theta) + a_3r^5cos^4\theta sin^2\theta}{1 - r\,cos\theta sin^2\theta + r^2(a_1cos^3\theta sin\theta + cos^2\theta sin^2\theta) + r^3(cos^5\theta - a_2cos^2\theta sin^3\theta) + a_3r^4cos^5\theta sin\theta}$$

Letting the above formula be Taylor expansion at $r = 0$, we can get

$$\frac{dr}{d\theta} = cos^2\theta sin\theta r^2 + (a_1cos^2\theta sin^2\theta - cos^3\theta sin\theta + cos^3\theta sin^3\theta)r^3 + etc$$
$$= R_2(\theta)r^2 + R_3(\theta)r^3 + etc. \qquad (3.7)$$

Letting $r(\theta, c) = r_1(\theta)c + r_2(\theta)c^2 + r_3(\theta)c^3 + r_4(\theta)c^4 + r_5(\theta)c^5 + etc.$ (3.8) become a solution to equation (3.7). And here by the initial conditions $r(0, c) = c,\ \ |c| \ll 1$.

Comparison coefficient

Substituting it into equation (3.8)

$$r(0, c) = r_1(0)c + r_2(0)c^2 + r_3(0)c^3 + r_4(0)c^4 + r_5(0)c^5 + hot = c$$

Comparing the coefficients on both sides of the equation, we can get

$$r_1(0) = 1, r_2(0) = r_3(0) = \cdots = 0. \qquad (3.9)$$

Substituting the power series of equation (3.8) into equation (3.7), the following identity is obtained

$$\begin{aligned} & r_1'(\theta)c + r_2'(\theta)c^2 + r_3'(\theta)c^3 + \cdots \\ & \equiv R_2(\theta)(r_1(\theta)c + r_2(\theta)c^2 + r_3(\theta)c^3 + \cdots)^2 + R_3(\theta)(r_1(\theta)c + r_2(\theta)c^2 + r_3(\theta)c^3 + \cdots)^3 \\ & \equiv cos^2\theta sin\theta(r_1(\theta)c + r_2(\theta)c^2 + r_3(\theta)c^3 + \cdots)^2 \\ & + (a_1cos^2\theta sin^2\theta - cos^3\theta sin\theta + cos^3\theta sin^3\theta)(r_1(\theta)c + r_2(\theta)c^2 + r_3(\theta)c^3 + \cdots)^3 \end{aligned}$$

Comparing the coefficients of the powers of c to obtain a series of differential equations as follows

$$r_1'(\theta) = 0$$
$$r_2'(\theta) = R_2(\theta)r_1(\theta)^2 = cos^2\theta sin\theta r_1(\theta)^2$$
$$r_3'(\theta) = R_3(\theta)r_1(\theta)^3 + 2R_2(\theta)r_1(\theta)r_2(\theta)$$
$$= (a_1cos^2\theta sin^2\theta - cos^3\theta sin\theta + cos^3\theta sin^3\theta)r_1(\theta)^3 + 2cos^2\theta sin\theta r_1(\theta)r_2(\theta)$$
$$\cdots\cdots\cdots$$

Then contact the initial conditional equation (3.9), we can find their solutions one by one

$$r_1(\theta) = 1$$
$$r_2(\theta) = \int_0^\theta R_2(\varsigma)\,d\varsigma$$

$$= \int_0^\theta cos^2\varsigma sin\varsigma \, d\varsigma = \frac{1}{3} - \frac{1}{3}cos^3\theta$$

$$r_3(\theta) = \int_0^\theta \{R_3(\varsigma) + 2R_2(\varsigma)r_2(\varsigma)\}\, d\varsigma$$

$$= \int_0^\theta \{(a_1cos^2\varsigma sin^2\varsigma - cos^3\varsigma sin\varsigma + cos^3\varsigma sin^3\varsigma) + 2cos^2\varsigma sin\varsigma r_2(\varsigma)\}\, d\varsigma$$

$$= -\frac{1}{4}a_1cos^3\theta sin\theta + \frac{1}{8}a_1cos\theta sin\theta + \frac{5}{18}cos^6\theta - \frac{2}{9}cos^3\theta + \frac{1}{8}a_1\theta - \frac{1}{18}$$

$$r_4(\theta) = \frac{1}{2}a_1cos^6\theta sin\theta - \frac{7}{20}a_1cos^4\theta sin\theta - \frac{1}{5}a_2cos^4\theta sin\theta - \frac{1}{4}a_1cos^3\theta sin\theta - \frac{1}{45}a_1cos^2\theta sin\theta + \frac{2}{5}a_2cos^2\theta sin\theta$$
$$+ \frac{1}{8}a_1cos\theta sin\theta - \frac{2}{45}a_2sin\theta - \frac{1}{5}a_2sin\theta - \frac{8}{27}cos^9\theta + \frac{5}{18}cos^6\theta - \frac{1}{12}a_1\theta cos^3\theta + \frac{1}{8}a_1\theta + \frac{1}{54}$$

$$\cdots\cdots\cdots$$

Because $R_2(\theta)$ is a continuous differentiable function with a period of 2π, here is

$$r_2(\theta) = g_2\theta + \varphi_2(\theta)$$
$$g_2 = \frac{1}{2\pi}\int_0^{2\pi} R_2(\varsigma)d\,\varsigma$$
$$\varphi_2(\theta) = \varphi_2(\theta + 2\pi)$$

If $g_2 = 0$, then $r_2(\theta) = \varphi_2(\theta)$,$r_2(\theta)$ is also a continuous differentiable function with a period of 2π. Calculate in turn, if $g_3 = 0, r_2(\theta)$ is a function of 2π cycles. Through the calculation of the above method, we can get

$$g_2 = 0$$
$$g_3 = \frac{1}{4}a_1\pi$$
$$g_4 = \frac{1}{12}a_1\pi$$
$$g_5 = \frac{1}{8}a_3\pi + \frac{3}{32}{a_1}^2\pi^2 + \frac{1}{16}a_1\pi$$
$$g_6 = \frac{1}{8}a_3\pi + \frac{7}{96}{a_1}^2\pi^2 + \frac{1}{144}a_1\pi$$
$$g_7 = \frac{163}{6144}{a_1}^3\pi + \frac{7}{96}{a_1}^3\pi^3 + \frac{3}{64}a_2\pi + \frac{17}{128}a_3\pi + \frac{11}{144}{a_1}^2\pi^2 + \frac{1}{8}a_1a_3\pi^2 + \frac{236}{13824}a_1\pi$$
$$\cdots\cdots\cdots$$

Use a successor function to balance point types

Define the successor function V as follows

$$\mathrm{V} \equiv \mathrm{r}(2\pi, \mathrm{c}) - \mathrm{r}(0, \mathrm{c})$$
$$= 2\pi g_3c^3 + 2\pi g_4c^4 + 2\pi g_5c^5 + 2\pi g_6c^6 + 2\pi g_7c^7 + \cdots$$

The following are discussed in four situations

(1)When $a_1 \neq 0$, then $g_3 \neq 0$, so $V \neq 0$, and the sign of the successor function V is determined by g_3, at this time

$$\mathrm{V} = \frac{1}{2}a_1\pi c^3 \begin{cases} a_1 > 0, V > 0, \\ a_1 < 0, V < 0. \end{cases}$$

So, When $a_1 > 0$, A(0,0) is the first-order unstable fine focus; when $a_1 < 0$, A(0,0) is the first-order stable fine focus.

(2) When $a_1 = 0, a_3 \neq 0$, then $g_3 = g_4 = 0, g_5 \neq 0$, so $V \neq 0$, and the sign of the successor function V is determined by g_5, at this time

$$\mathrm{V} = \frac{1}{4}a_3\pi c^3 \begin{cases} a_3 > 0, V > 0, \\ a_3 < 0, V < 0. \end{cases}$$

So when $a_1 = 0, a_3 > 0$, A(0,0) is the second-order unstable fine focus; when $a_1 = 0, a_3 < 0$, A(0,0) is the second-order stable fine focus.

(3) When $a_1 = a_3 = 0, a_2 \neq 0$, then $g_3 = g_4 = g_5 = g_6 = 0, g_7 \neq 0$, so $V \neq 0$, and the sign of the successor function V is determined by g_7, at this time

$$V = \frac{3}{32} a_2 \pi c^7 \begin{cases} a_2 > 0, V > 0, \\ a_2 < 0, V < 0. \end{cases}$$

So when $a_1 = a_3 = 0, a_2 > 0$, A(0,0) is the third-order unstable fine focus; when $a_1 = a_3 = 0, a_2 < 0$, A(0,0) is the third-order stable fine focus.

(4)When $a_1 = a_2 = a_3 = 0$, at this time, the original system is

$$\begin{cases} \frac{dx}{dt} = y + xy + x^2 y \equiv P(x,y), \\ \frac{dy}{dt} = -x \equiv Q(x,y). \end{cases}$$

At this time, the vector field $\mathrm{P(x,y)}$ and $\mathrm{Q(x,y)}$ defined by the function on the right side of the system are symmetric about the x-axis, that is

$$P(x,-y) = P(x,y)$$
$$Q(x,-y) = Q(x,y)$$

So when $a_1 = a_2 = a_3 = 0$, A(0,0) is the center of the system.

4. The existence and number of limit cycles

Theorem 3:

Case 1: If $a_1 \neq 0$, then a first-order hopf bifurcation occurs at $\delta = 0$, and the parameter region in which the system has a unique limit cycle in the vicinity is

$$a_1\delta < 0, 0 < |\delta| \ll |a_1| \ll 1$$

Case 2: If $a_3 \neq 0$, then a second-order hopf bifurcation occurs at $a_1 = 0,\ \delta = 0$, and the parameter region where the system has two limit cycles near the original is

$$a_1 a_3 < 0, a_3\delta > 0, 0 < |\delta| \ll |a_1| \ll |a_3| \ll 1$$

Case 3: If $a_2 \neq 0$, then a third-order hopf bifurcation occurs at $a_1 = 0, a_2 = 0, \delta = 0$, and the parameter region where the system has three limit cycles in the vicinity of the original is

$$a_2 a_3 < 0,\ a_1 a_3 > 0,\ a_1\delta < 0, 0 < |\delta| \ll |\, a_1| \ll |\, a_3| \ll |\, a_2| \ll 1$$

Proof As Follows:

(1) When $a_1 > 0, \delta = 0$, known by theorem 2, A(0,0) is the first-order unstable fine focus, when δ decreases from zero, at this time, A(0,0) changes from the first-order unstable fine focus to the stable coarse focus, the singularity is caused by the absorption of energy to the release of energy. In this process, equal amplitude oscillations must occur, according to the hopf branch theory, an unstable limit cycle is generated at this time. When $a_1 < 0, \delta = 0$, known by theorem 2, A(0,0) is the first-order stable fine focus, when δ increases from zero, at this time, A(0,0) changes from the first-order stable fine focus to the unstable coarse focus, the singularity from releasing energy to absorbing energy. In this process, equal amplitude oscillations must occur, according to the hopf branch theory, an unstable limit cycle is generated at this time. And the parameter area where the system has a unique limit cycle near the origin is

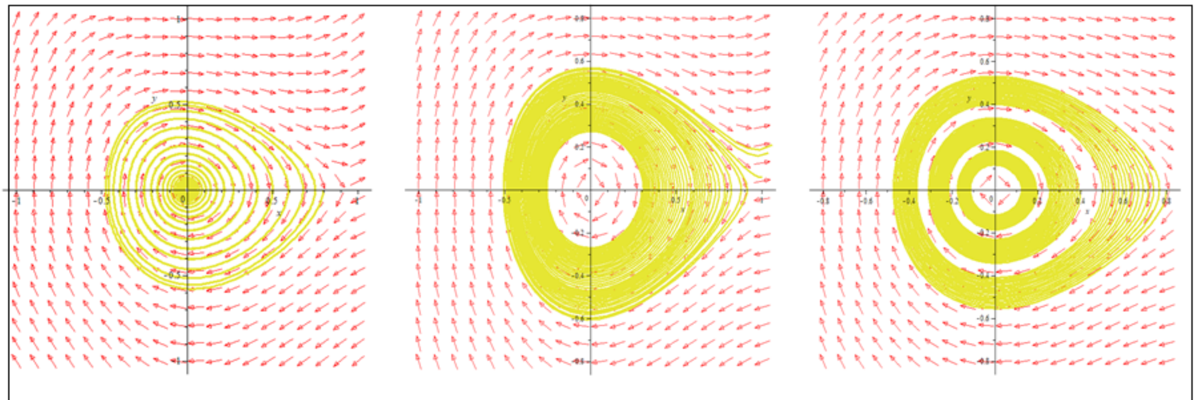
$$a_1\delta < 0, 0 < |\delta| \ll |a_1| \ll 1$$

(2) When $a_3 > 0, a_1 = \delta = 0$, known by theorem 2, A(0,0) is the second-order unstable fine focus, when a_1 decreases from zero, A(0,0) is changed from a second-order unstable fine focus to a first-order stable fine focus, from the hopf branch theory, a limit cycle is generated at this time; when δ increases from zero, at this time, A(0,0) changes from the first-order stable fine focus to the unstable coarse focus, at this point, a limit cycle is generated. The entire process produces two limit cycles. When $a_3 < 0, a_1 = \delta = 0$, known by theorem 2, A(0,0) is the second-order stable fine focus, when a_1 decreases from zero, A(0,0) is changed from a second-order stable fine focus to a first-order unstable fine focus, from the hopf branch theory, a limit cycle is generated at this time; when δ decreases from zero, at this time, A(0,0) changes from the first-order unstable fine focus to the stable coarse focus, at this point, a limit cycle is generated. The entire process produces two limit cycles. And the parameter region where the system has two limit cycles near the original is

$$a_1 a_3 < 0, a_3\delta > 0, 0 < |\delta| \ll |a_1| \ll |a_3| \ll 1$$

(3) This situation can be discussed similarly.

Numerical Simulation：



(a) $a_1 = 0.5, \delta = -0.07$ (b) $a_1 = -0.1, a_3 = 0.5,, \delta = 0.01$ (c) $a_1 = 0.1,\ a_2 = 0.6,\ a_3 = -0.3, \delta = -0.01$

CONCLUSIONS

In this paper, we use the classical method of conventional differential equation qualitative theory to qualitatively analyze a class of planar fifth-order polynomial systems, and use the successor function method to analyze the fine focus of the system. The principle of symmetry is used to determine the center of the system and uses the Hopf branch theory. The existence and number of limit cycles under different parameters are analyzed, and the parameter regions with one, two and three limit cycles are obtained.

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