Motion of a Batted Ball for Striking a Boundary in a Cricket Game

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Abstract	Original Research Article

This is a model for motion of a ball hit by a batsman to score a boundary. The batted ball hits the ground, bounces several times and then grazes the ground to reach the boundary line evading intercept by any fieldsman. With given initial velocity, time taken by the batted ball to cross the boundary line covering the distance between the batsman and the ball touching the boundary line is determined. Herein is also determined the minimum initial velocity of the batted ball and its corresponding direction to strike a "four" .Some numerical examples are also cited.
Keywords: Cricket Game, Motion, Batted Ball, batsman, boundary.

INTRODUCTION

SN Maitra [1] earlier innovated two models of projectile motion of a cricket Ball for 'bowled out' and 'caught out' respectively. In this design is modeled a projectile motion of a cricket ball crashed by a batsman followed by its bouncing motion on the ground and thereafter rectilinear motion grazing the ground till it crosses or touches the boundary line, without being stopped by any fielder.

EQUATIONS OF MOTION OF BATTED BALL IN AIR

Let the batsman play a shot and the ball leave the bat with a velocity u downwards at angle α to the horizontal. If ball strikes the ground descending a height h and describing a horizontal distance R_0 in time T_0, escaping any intervention by a fieldsman, then its equations of motion in the air, whose resistance is neglected and where g is the acceleration due to gravitation, are given by

$$h = (u \sin\alpha)T_0 + \frac{1}{2} g{T_0}^2 \qquad (1)$$

$$R_0 = u \cos\alpha . T_0 \qquad (2)$$

Eliminating T_0 between (1) and (2) we can find

$$R_0 = (u/g)(-u \sin\alpha + \sqrt{u^2\sin^2\alpha + 2gh})\cos\alpha \qquad (3)$$

$$T_0 = (-u\sin\alpha + \sqrt{u^2\sin^2\alpha + 2gh})/g \qquad (4)$$

Equations of motion of the batted ball on the ground

Overall motion of the batted ball discussed in this section consists of its motion from the batsman to the hitting ground and its successive bounces from the ground followed by its rectilinear motion on the ground till it reaches the boundary line to credit the batsman with a boundary, ie, four runs. Let the ball played by the batsman strike the ground with velocity v at angle β to the horizontal, the coefficient of elasticity between the ball and the ground being e, then

$$v^2\sin^2\beta = u^2\sin^2\alpha + 2gh \qquad (5)$$

because of gravity in the vertical direction which is also the line of impact, the horizontal component of the velocity of the ball is constant so that

$$u\cos\alpha = v \cos\beta \qquad (6)$$

Case1. Theoretically the ball executes many rebounds, so to say infinitely many rebounds before its vertical component of the velocity vanishes. By Newton's law of collision the vertical component of the velocity after first

rebound from the ground is evsinβ. Further in view of the velocity of rise being equal to the velocity of fall the subsequent vertical components of velocities of the ball due to successive rebounds, say, up to the nth rebound are given by/

$$ev\sin\beta, e^2v\sin\beta,\ e^3v\sin\beta\ \dots\dots\dots \qquad\qquad e^nv\sin\beta \qquad\qquad (7)$$

Since the time of fall is equal to the time of rise,the total time elapses up to the nth rebound is given by

$$T'=2(ev\sin\beta+e^2v\sin\beta+e^3v\sin\beta\ \dots\dots\dots \qquad\qquad +\ e^nv\sin\beta)/g \qquad (\text{vide G.P. series})$$
$$=\frac{2e}{g}\cdot\frac{1-e^n}{1-e}v\sin\beta \qquad (\text{By use of (5)})$$
$$=\frac{2e}{g}\cdot\frac{1-e^n}{1-e}\sqrt{u^2\sin^2\alpha+2gh} \qquad\qquad (8)$$

whereas the total distance described along the ground in this time is obtained as
$$R'=(u\cos\alpha)\frac{2e}{g}\cdot\frac{1-e^n}{1-e}\sqrt{u^2\sin^2\alpha+2gh} \qquad\qquad (9)$$

If the horizontal distance from the just-batted ball of the boundary line compatible with motion of the smashed ball that reaches/crosses the boundary line after completion of n bounces be S_1 to score a boundary,

$$R_0+R\geq S_1 \qquad\qquad (10)$$

Case2.While watching a cricket matches, it is observed that the smitten ball moves on bouncing, stops bouncing and then moves on the ground in a rectilinear path to reach/cross the boundary line for 'four'. Before the ball ceases to any further bounce, there arises a textbook problem of Dynamics[3], giving infinitely many rebounds such that in consideration of (8) and (9) it covers a horizontal distance R_1 in time T_1 due to bouncing:

$$R_1=\lim n_{\to\infty}\frac{2eu}{g}\cdot\frac{1-e^n}{1-e}\sqrt{u^2\sin^2\alpha+2gh}\ \cos\alpha \qquad\qquad (11)$$
$$=\frac{2eu}{g}\cdot\frac{1}{1-e}\sqrt{u^2\sin^2\alpha+2gh}\ \cos\alpha \quad (e<1,e^n\to 0\ \text{as}\ n\to\infty)$$

$$T_1=\lim_{n\to\infty}\frac{2e}{g}\cdot\frac{1-e^n}{1-e}\sqrt{u^2\sin^2\alpha+2gh}$$
$$=\frac{2e}{g}\cdot\frac{1}{1-e}\sqrt{u^2\sin^2\alpha+2gh} \qquad\qquad (12)$$

After the ball stops bouncing, it begins to move on the ground obviously towards the boundary line, of course, if any fieldsman is unable to stop it. Let f be the frictional resistance of the ground per unit mass of the ball that crosses the boundary line with velocity u_0 traveling in this context a distance R_2 in time T_2:

$$R_2=\frac{u^2\cos^2\alpha-u_0^2}{2f} \qquad\qquad (13)$$

$$T_2=\frac{u\cos\alpha-u_0}{f} \qquad\qquad (14)$$

Thus in consonance with (3),(4),(11),(12),(13) and (14) the total horizontal distance traveled by the batted ball and the time reckoned from the instant of striking the ball by the batsman are given by

$$S=R_0+R_1+R_2=[(-u\sin\alpha+\in\sqrt{u^2\sin^2\alpha+2gh})/g+\frac{u^2\cos^2\alpha-u_0^2}{2fu\cos\alpha}]\ u\cos\alpha \qquad (15)$$

where $\in=\frac{1+e}{1-e}$ \qquad\qquad (15.1)

$$T=T_0+T_1+T_2=(-u\sin\alpha+\in\sqrt{u^2\sin^2\alpha+2gh})/g+\frac{u\cos\alpha-u_0}{f} \qquad\qquad (16)$$
Combining (15) and (16) in some way one gets for a boundary

$$S=(u\cos\alpha)(T+\frac{u_0}{f})-\frac{u^2\cos^2\alpha+u_0^2}{2f} \qquad\qquad (17)$$

Now let us express the batted velocity, ie, exit velocity u of the ball from the contact with the bat in terms of the distance S:

$$\frac{g(S+\frac{u_0^2}{2f})}{(u\,cos\alpha)} +u\,sin\alpha= \in \sqrt{u^2 sin^2\alpha + 2g\,h} +\frac{g}{2f}u\,cos\alpha$$

Or, $\{\frac{g\left(S+\frac{u_0^2}{2f}\right)}{(u\,cos\alpha)} + u(sin\alpha - \frac{g}{2f}\,cos\alpha)\}^2=\in^2 (u^2 sin^2\alpha + 2gh),$

which, because of substitutions

A=$\frac{g(S+\frac{u_0^2}{2f})}{(cos\alpha)}$,B= $sin\alpha - \frac{g}{2f}\,cos\alpha$,$C^2=\in^2 sin^2\alpha$ and D=2 $\in^2$gh   (18)

 turns out to be
$(\frac{A}{u} + Bu)^2 =C^2 u^2 + D$
Or,$(C^2 - B^2)u^4 - (2AB - D)u^2 - A^2 = 0$,   C>B, 2AB>D ,$\in> 1$
$mu^4 - 2pu^2 - A^2 = 0$
u=$\sqrt{(\frac{p+\sqrt{p^2+mA^2}}{m})}$ (19)
where $C^2 - B^2 = m, 2AB - D = 2p$ (19.1)

Maximum Distance Covered by the Ball during Successive Bounces
        In this section are determined the maximum distance the ball can describe on the ground while it is bouncing 'up and down' and the optimum angle of striking to the horizontal by the batsman. Then with given initial batted velocity u, to crack a boundary, recalling (10) and (11), the following inequality holds

$R_0+(R_1)$max$\geq S_1$ (20)

However for maximum or minimum of R_1
$\frac{dR_1}{d\alpha}=0$ (21)

Or, equivalently $\frac{d(R_1^2)}{(cos2\alpha)}=0$ (22)

and as such (11) is rewritten as
$\frac{R_1^2 g^2}{4e^2}(1-e)^2=\frac{u^4}{4}sin^2 2\alpha + ghu^2(1+cos2\,\alpha)$ (23)

$=\frac{u^4}{4}(1-cos^2 2\alpha) + ghu^2(1+cos2\,\alpha)$

so that by use of (22) one gets
$cos2\,\alpha = \frac{2gh}{u^2}$ $cos\alpha_{opt} =\frac{1}{2}cos^{-l}(\frac{2gh}{u^2})$ (24)

Or, $cos\alpha_{opt} =\sqrt{\frac{1}{2}(1 + 2\frac{gh}{u^2})}$ $sin\alpha_{opt} =\sqrt{\frac{1}{2}(1 - 2\frac{gh}{u^2})}$ (25)
$\frac{d^2 R_1^2}{d(cos2\,\alpha)^2}<0$ from (23)

and hence by use of (11),(12) and (25),the maximum horizontal distance covered by bounces is obtained as
$(R_1)_{max}=\frac{e(u^2+2gh)}{g(1-e)}$ (26)

in time $T_{1\,opt}=\frac{2e}{g(1-e)}\sqrt{\frac{u^2+2gh}{2}}$ (26.1)

with fixed height h and initial velocity u.

Minimum Batted Velocity for a given horizontal distance to be covered by Bounces

We can show that with a fixed distance R_1 on the ground to be described by bounces of batted ball having its initial height h, there exist a minimum velocity of the batted ball and the corresponding angle of projection. So from equation (23), for maximum or minimum of u, ie, u^2 we get

$$\frac{d(u^2)}{(cos2\alpha)} = 0 \tag{27}$$

Hence differentiating (23) and using (27), one gets

$$ghu^2 - \frac{u^4}{4}(2cos2\alpha) = 0$$

$$u_{min}^2 = \frac{2gh}{cos2\alpha_{opt}} \tag{28}$$

Now differentiating (23) twice with respect to $(cos2\alpha)$ subject to (27)we *obtain*

$$[\frac{u^2}{2}\{1-(cos2\,\alpha)^2\} + gh(1 + cos2\alpha)]\,\frac{d^2(u^2)}{d(cos2\,\alpha)^2} = \frac{u^2}{2}$$

which implies $\frac{d^2(u^2)}{d(cos2\,\alpha)^2} > 0 \tag{29}$

which ratifies the minimum velocity given by (28):

To determine u_{min} and α_{opt} explicitly we employ (28) in (23):

$$\frac{R_1^2 g^2}{4e^2}(1 - e)^2 = u^2(1 + cos2\,\alpha)\{\frac{u^2}{4}(1 - cos2\alpha) + gh\}$$

$$= u^2\,(1 + \frac{u^2}{2gh})\,\{\frac{u^2}{4}(1 - \frac{u^2}{2gh}) + gh\}$$

$$= (u^2 + 2gh)^2/4$$

$$\frac{R_1 g(1-e)}{e} = u^2 + 2gh$$

$$u_{min} = \sqrt{\frac{g}{e}\{R_1(1 - e) - 2he\}} \tag{30}$$

which in consequence of (28) gives

$$cos2\alpha_{opt} = \frac{2he}{R_1(1-e)-2he} \tag{31}$$

Or, $cos^2\alpha_{opt} = \frac{R_1(1-e)}{2\{R_1(1-e)-2he\}}$

$$\alpha_{opt} = cos^{-|}\sqrt{\frac{R_1(1-e)}{2\{R_1(1-e)-2he\}}} \tag{32}$$

which suggests that if h=0 or $h \to 0, \alpha_{opt} = 45^0\,or\,\alpha_{opt} \to 45^0$ then it reduces to a textbook problem. It is observed that with fixed batted velocity the maximum bouncing-horizontal- distance or with with fixed bouncing-horizontal-distance the minimum batted velocity can be determined from either of equations (26) and (30).

Time taken by Batted Ball to reach the Boundary

Eliminating u_0 between (15) and (16) the time taken to reach the boundary line by the ball slapped by the batsman is given by

$$T = (-usin\alpha + \in \sqrt{u^2 sin^2\alpha + 2gh}\,)\frac{1}{g} + \frac{1}{f}[ucos\alpha - \{\frac{u^2}{2f}cos^2\alpha - S - (-usin\alpha + \in\sqrt{u^2sin^2\alpha + 2gh}\,)\frac{ucos\alpha}{g}\}^{\frac{1}{2}}.\sqrt{2f}\,]$$

$$e < 1, \in > 1 \tag{33}$$

From (33) *it* is ascertained that greater is the horizontal component of the batted velocity, is less the time for the batted ball to reach the boundary .

Further if $\alpha \to 0$, ie, the initial batted velocity is in the horizontal direction, then the time to reach the boundary is

$$T_3 = \frac{\in}{g}\sqrt{2gh} + \frac{1}{f}[u - \{\frac{u^2}{2f} - S - \frac{\in}{g}\sqrt{2gh}.u\}^{1/2}\sqrt{2f}\,] \tag{34}$$

7. *M*inimum Initial batted Velocity to strike a 'Four':

At the sight of equation (15) or its another form, with given initial angle $\alpha,$ the batted velocity depends upon $h, S, \in$ and $g.$ Nevertheless intuitively there exists angle α_{opt} of projection by the batsman to achieve a 'Four' with a minimum batted velocity of the ball, for which we need to put $\frac{du}{d\alpha} = 0$ from (15) and then to find the corresponding value of $\alpha.$ But this gives a complicated equation involving $\sin\alpha$ and $\cos\alpha.$ In order to avoid such complication we neglect h because h<<S in equation (15) which is rewritten as

$\frac{A}{u^2} = (\in -1)\frac{\sin2\alpha}{2} + \frac{\lambda}{2}(1+\cos2\ \alpha) + \frac{\sqrt{2gh}}{2u}(1+\cos2\ \alpha) \in$

(Writing $\sqrt{u^2\sin^2\alpha + 2gh} \cong u\sin\alpha + \sqrt{2gh}\cos\alpha \leq \sqrt{u^2\sin^2\alpha + 2gh}$)

Or, $\frac{2A}{u^2} = (\in -1)\sin2\alpha + (1 + \cos2\ \alpha)(\lambda + \in \sqrt{2gh}\ /u)$ \qquad (35)

where $A = g(S + \frac{u_0^2}{2f}), \lambda = \frac{g}{2f}$ \qquad (36)

For minimum or maximum of u, ie, for maximum or minimum of $\frac{1}{u^2}$ we have from Equation (35)

$\frac{d(\frac{1}{u^2})}{d\alpha} = 0$ \qquad (37)

Or, $(\in -1)\cos2\alpha - (\lambda + \in \frac{\sqrt{2gh}}{u_{min}})\sin2\alpha = 0$

Or, $\tan2\alpha = \frac{(\in -1)}{\lambda + \in \frac{\sqrt{2gh}}{u_{min}}} = \mu$ \qquad (38)

Ultimately to evaluate u_{min} and α_{opt} with desired accuracy a method of approximation is adopted. Since h<<S implying $\in \frac{\sqrt{2gh}}{u_{min}} \ll \lambda$, relation (38) gives

$\tan2\alpha_{opt} = \frac{(\in -1)}{\lambda}$ \qquad (39)

so that $\sin2\alpha = \frac{\mu}{\sqrt{1+\mu^2}}$, $\cos2\alpha = \frac{1}{\sqrt{1+\mu^2}}$ \qquad (40)

which on substitution into (35) yields

$u_{min}^2 = \frac{2\sqrt{1+\mu^2}A}{(\in -1)\mu + \left(\lambda + \in \frac{\sqrt{2gh}}{u_{min}}\right)(1+\sqrt{1+\mu^2})}$ \qquad (41)

Denoting $(\in -1)\mu + \lambda\left(1 + \sqrt{1 + \mu^2}\right), \in \left(1 + \sqrt{1 + \mu^2}\right)$ and $2A\sqrt{1 + \mu^2}$ by

E, 2F and G respectively (41) reduces to a quadratic equation

$Eu_{min}^2 + 2\sqrt{2gh}\ Fu_{min} - G = 0$ \qquad (42)

whose solution gives

$u_{min} = \frac{-\sqrt{2gh}\ F + \sqrt{2ghF^2 + GE}}{E}$ \qquad (43)

which involves h, however small it is in comparison to S.

Substituting (41) into (38), we can obtain more accurate value of α_{opt} .

Numerical Examples

Rearranging (15) we can find the velocity u_0 with which the batted ball can reach the boundary line and using (16) the time taken to reach it.

$u_0 = [\left(-u\sin\alpha + \in \sqrt{u^2\sin^2\alpha + 2gh}\right)\frac{u\cos\alpha}{g} + \frac{u^2}{2f}cos^2\alpha - S]^{1/2}\sqrt{2f}$ \qquad (44)

Example1. Let us suppose the initial velocity of the batted ball =u=25meter per second, ie, 90 kilometer per hour. Radius S of the cricket ground=65 meters and g=10 meter/sec^2 . Then from (44) and (16) with some realistic

values of $\in$ and α, the velocity with which the ball passes the boundary line $= u_0$=22.47meter/sec in timeT=3.138 seconds.

But in case of the lifted batted- ball at angle

α above the horizontal line, in the foregoing equations α is to be replaced by $-\alpha$.

Example2. With u=20 meter/sec ie 72kms/hour, similarly, u_0=11.97meter/sec, T=4.57seconds.

Example3. With u=30 meter/second,$ie, \frac{72km}{hour}$, similarly u_0=27.47,T=2.58seconds.

Example 4.With u=35 meter/sec ,ie, 126km/hr, similarly u_0=2 meter/secs, T=4.16seconds.

Figure-1.Three different batsmen smash the balls for boundary

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