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The Summation Problem of Linear Recursive Series

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Abstract **Review Article**

In this paper, the summation of the first and second order linear recursive sequences is discussed. The general methods of summation of these two sequences and some applications in summation are given.
Keywords: recursive sequence; general term formula; Sequence summation.

INTRODUCTION

Linear recursive sequence is a kind of sequence whose general term is given by recurrence formula. The key to solve the problem of summation of linear recursive sequence is to obtain the general term formula of the sequence and convert it into the summation of equal difference or equal ratio sequence. This paper mainly discusses the summation method of first and second order linear recursive sequence.

Summation of first order linear recursive sequence

Definition 1: From recurrence formula

$a_{n+1} = pa_n + d$ （p, d is Constant and $p \neq 0$, a_1 is known, $n \in N_+$）, (1)

The defined sequence $\{a_n\}$ is called first order linear recursive sequence.

When $p = 1$ or $d = 0$, $\{a_n\}$ is equal difference or equal ratio sequence, the sum of $\{a_n\}$ can be obtained from the summation formula of equal difference and equal ratio sequence.

When $p \neq 1$, let $\beta = \dfrac{d}{1-p}$, we can change (1) into $a_{n+1} - \beta = p(a_n - \beta)$ and $\{a_n - \beta\}$ is an equal ratio sequence with p as the common ratio, we can obtain

$$a_n = \beta + (a_1 - \beta)p^{n-1}, \quad (2)$$

Thus the sum can be obtained.

Example 1: Known $a_1 = \dfrac{3}{2}$, $a_{n+1} = \dfrac{1}{4}a_n + \dfrac{1}{2^{n+1}}$ $(n \geq 2)$, find $\sum_{n=1}^{\infty} a_n$.

Solution: Transform recurrence formula to

$$2^{n+1}a_{n+1} = \frac{1}{2}(2^n a_n) + 1,$$

sign $b_n = 2^n a_n$, know $\{b_n\}$ is a first-order linear recursive sequence, from $p = \frac{1}{2}$, $d = 1$, obtaining $\beta = \frac{d}{1-p} = 2$, according to formula (2), having

$$b_n = \beta + (b_1 - \beta) p^{n-1} = 2 + (2a_1 - 2)\frac{1}{2^{n-1}} = 2 + \frac{1}{2^{n-1}},$$

Solution is $a_n = \frac{1}{2^{n-1}} + \frac{1}{2} \cdot \frac{1}{4^{n-1}}$, When $r = \frac{1}{2}$ or $r = \frac{1}{4}$ is noted, we all have $|r| < 1, \lim_{n\to\infty} r^n = 0$, obtaining

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} \left[\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} + \frac{1}{2} \cdot \frac{1-(\frac{1}{4})^n}{1-\frac{1}{4}} \right] = \frac{1}{1-\frac{1}{2}} + \frac{1}{2} \cdot \frac{1}{1-\frac{1}{4}} = \frac{8}{3}.$$

Summation of second order linear recursive sequence

Definition 2: From recurrence formula

$$a_{n+2} = p_1 a_{n+1} + p_2 a_n + p_3$$

(p_1, p_2, p_3 is Constant and $p_2 \neq 0, a_1, a_2$ is known, $n \in N_+$) (3)

The defined sequence $\{a_n\}$ is called second order linear recursive sequence.

In particular, when $p_3 = 0$, $\{a_n\}$ is also called second order linear homogeneous recursive sequence?

Before discussing the summation of the second order linear homogeneous recursive sequence, a general term formula of the sequence is proved.

General term formula: If the general term of second order linear homogeneous recursive sequence $\{a_n\}$ is determined by $a_{n+2} = p_1 a_{n+1} + p_2 a_n$, then

(1) When $\lambda^2 - p_1\lambda - p_2 = 0$ has two different (real or complex) roots λ_1 and λ_2, there is a general term formula

$$a_n = c_1 \lambda_1^{n-1} + c_2 \lambda_2^{n-1} \qquad (4)$$

Where c_1, c_2 is uniquely determined by the two equations $c_1 + c_2 = a_1$ and $c_1\lambda_1 + c_2\lambda_2 = a_2$.

(2) When $\lambda^2 - p_1\lambda - p_2 = 0$ has double root λ, there is a general term formula

$$a_n = (c_1 + nc_2)\lambda^{n-1} \qquad (5)$$

Where c_1, c_2 is uniquely determined by the two equations $c_1 + c_2 = a_1$ and $c_1\lambda_1 + c_2\lambda_2 = a_2$.

Prove (1) Proving by mathematical induction.

When $n = 3$, by $c_1 + c_2 = a_1$, $c_1\lambda_1 + c_2\lambda_2 = a_2$, λ_1、λ_2 satisfing $\lambda^2 - p_1\lambda - p_2 = 0$, having

$$a_3 = p_1a_2 + p_2a_1 = p_1(c_1\lambda_1 + c_2\lambda_2) + p_2(c_1 + c_2)$$
$$= c_1(p_1\lambda_1 + p_2) + c_2(p_1\lambda_2 + p_2) = c_1\lambda_1^2 + c_2\lambda_1^2,$$

When you know $n = 3$, the conclusion is true.

Suppose when $n = k$, having $a_k = c_1\lambda_1^{k-1} + c_2\lambda_2^{k-1}$, then when $n = k+1$, having

$$\begin{aligned} a_{k+1} &= p_1a_k + p_2a_{k-1} = p_1(c_1\lambda_1^{k-1} + c_2\lambda_2^{k-1}) + p_2(c_1\lambda_1^{k-2} + c_2\lambda_2^{k-2}) \\ &= p_1c_1\lambda_1^{k-1} + p_2c_1\lambda_1^{k-2} + p_1c_2\lambda_2^{k-1} + p_2c_2\lambda_2^{k-2} \\ &= c_1\lambda_1^{k-2}(p_1\lambda_1 + p_2) + c_2\lambda_2^{k-2}(p_1\lambda_2 + p_2) \\ &= c_1\lambda_1^{k-2}\cdot\lambda_1^2 + c_2\lambda_2^{k-2}\lambda_2^2 = c_1\lambda_1^k + c_2\lambda_2^k, \end{aligned}$$

That is to say, when $n = k+1$, the conclusion is also true. Therefore, according to mathematical induction, conclusion（1）is proved.

（2）The same method can prove the conclusion.

According to the above general formula, the summation of the second order linear homogeneous recursive sequence can be reduced to the summation of the equal ratio sequence or $\{n\lambda^{n-1}\}$.

Example 2 Let two adjacent terms of $\{a_n\}$ is a_n, a_{n+1}, which is two roots of equation $x^2 - c_nx + \frac{1}{9^n} = 0$, and $a_1 = 1$, finding $\sum_{n=1}^{\infty} c_n$.

Solution: When $n = 1$ and $a_1a_2 = \frac{1}{9}$, $a_2 = \frac{1}{9a_1} = \frac{1}{9}$ is obtained, by $a_na_{n+1} = \frac{1}{9^n}$ and $a_{n+1}a_{n+2} = \frac{1}{9^{n+1}}$, obtaining $a_{n+2} = \frac{1}{9}a_n$, so $\{a_n\}$ is a second-order linear homogeneous recursive sequence.

$\lambda_1 = \frac{1}{3}$, $\lambda_2 = -\frac{1}{3}$ are obtained by $\lambda^2 - \frac{1}{9} = 0$, to solve the system of equations

$$\begin{cases} c_1 + c_2 = a_1 \\ c_1\lambda_1 + c_2\lambda_2 = a_2 \end{cases} \text{ which is } \begin{cases} c_1 + c_2 = 1 \\ \frac{1}{3}c_1 - \frac{1}{3}c_2 = \frac{1}{9} \end{cases}, \text{ obtaining } \begin{cases} c_1 = \frac{2}{3} \\ c_2 = \frac{1}{3} \end{cases},$$

So by formula (4), having

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(c_1\lambda_1^{n-1} + c_2\lambda^{n-2}) = c_1\frac{1}{1-\lambda_1} + c_2\frac{1}{1-\lambda_2} = \frac{2}{3}\cdot\frac{3}{2} + \frac{1}{3}\cdot\frac{3}{4} = \frac{5}{4},$$

From $c_n = a_n + a_{n+1}$ obtaining $\sum_{n=1}^{\infty} c_n = 2\sum_{n=1}^{\infty} a_n - a_1 = 2\times\frac{5}{4} - 1 = \frac{3}{2}$.

Example 3: Known $a_{n+2} = a_{n+1} - \frac{1}{4}a_n$ and $a_1 = a_2 = 1$, finding $\sum_{k=1}^{n} a_k$.

Solving Equation $\lambda^2-\lambda+\frac{1}{4}=0$ has double roots $\lambda=\frac{1}{2}$, Solving equations $\begin{cases} c_1+c_2=a_1 \\ c_1\lambda+2c_2\lambda=a_2 \end{cases}$, which is $\begin{cases} c_1+c_2=1 \\ \frac{1}{2}c_1+c_2=1 \end{cases}$, obtaining $\begin{cases} c_1=0 \\ c_2=1 \end{cases}$, by formula (5), having

$$a_n=(c_1+nc_2)\lambda^{n-1}=n\lambda^{n-1}$$, signing $S_n=a_1+a_2+\cdots+a_n$, so

$$S_n-\lambda S_n=(1+2\lambda+\cdots+n\lambda^{n-1})-(\lambda+2\lambda^2+\cdots+n\lambda^n)=1+\lambda+\cdots+\lambda^{n-1}-n\lambda^n$$
$$=\frac{1-\lambda^n}{1-\lambda}-n\lambda^n,$$

Solution is $S_n=\sum_{k=1}^{n}k\lambda^{k-1}=\frac{1-\lambda^n}{(1-\lambda)^2}-\frac{n\lambda^n}{1-\lambda}=\frac{1-\frac{1}{2^n}}{(1-\frac{1}{2})^2}-\frac{n\cdot\frac{1}{2^n}}{1-\frac{1}{2}}=4(1-\frac{1}{2^n})-\frac{n}{2^{n-1}}$,

In formula (3), when $p_3\neq 0$, if $p_1+p_2=1$, formula (3) can be changed to

$$a_{n+2}-a_{n+1}=-p_2(a_{n+1}-a_n)+p_3,$$

So $\{a_{n+1}-a_n\}$ is a first-order linear recursive sequence. According to the discussion in the first part, we can get the general term formula $a_{n+1}-a_n=f(n)$ (known $f(n)$, Sum the two ends of the above formula to get the general formula of $\{a_n\}$, that is $a_n=a_1+\sum_{k=1}^{n-1}f(k)$, From this we can find the sum of series $\{a_n\}$; If $p_1+p_2\neq 1$, let $\beta=\frac{p_3}{1-p_1-p_2}$, change formula (3) to

$a_{n+2}-\beta=p_1(a_{n+1}-\beta)+p_2(a_n-\beta)$, It is known that $\{a_n-\beta\}$ is a second-order linear homogeneous recursive sequence. According to (4) or (5), the general term and its sum of $\{a_n\}$ can also be obtained.

So far, we have obtained the general method of finding the sum of the first and second order linear recursive sequences. It is worth noting that the general term formula of the first and second order linear recursive sequence is also helpful to discuss other properties of the sequence.

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