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Ineraction between Kepler's Law and Inverse-Square Law Sn Maitra

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Abstract **Review Article**

Combining Kepler's second law with the Universal law of gravitation in case of an elliptic orbit of a satellite around the Earth, Kepler's third law is derived. Given the elliptic orbit of a satellite it is proved that its motion is governed by the inverse-square law of gravitation and its orbital elements subject to the initial conditions are determined. The time period of the elliptical orbit is determined applying Inverse –square law without involving Kepler's law. However, this is done by evaluating a definite integral with a technique other than that available in the relevant textbooks. While closing, two examples demonstrating Inverse –square law of attraction and repulsion, ie, Coulomb's law have been worked out for the sake of pedagogy.

Keywords: Ineraction, Kepler's Law, Inverse-Square Law, pedagogy.

INTRODUCTION

1. Umapati Patter [1] reproduced Kepler's laws and Inverse-Square law of gravitation with brief historical background and referred to as elliptical orbits of planets around the Sun redefining the orbital elements like major axis, minor axis, eccentricity, perihelion, aphelion etc. He[1] mentioned polar equation of an ellipse, expressions for radial and tranverse components of the acceleration of a particle in two-dimensional curvilinear motion. Following Kepler's second law, he[1] obtained constant angular momentum of a planet/particle in elliptic orbit, which obviously suggests zero transverse acceleration but did not prove vividly that the radial acceleration is inversely proportional to the square of its distance from the centre of force vis-à-vis the Sun in case of a planet and is always directed to the latter. Relation [4] leaves much to be desired in as much as this aspect, ie, Inverse-square law of force for the elliptic orbit has to be developed by use of polar equation of the ellipse, as well as the expressions for the radial and transverse accelerations [1-3]. However from pedagogic view point this is done in Section2 as follows:

2. Derivation of Inverse-Square law of Gravitation for Elliptic Orbit of a Satellite/Planet

We rewrite the formulae of radial and transverse velocities and accelerations along with the equation of the elliptic orbit of planet[1]/ selellite[1], respectively

$$v_r = \frac{\mathrm{dr}}{\mathrm{dt}},\ v_\theta = \mathrm{r}\frac{\mathrm{d}\theta}{\mathrm{dt}} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$f_r = \frac{d^2r}{dt^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{dt}}\right)^2,\ f_T = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{dt}}\right) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\frac{1}{r} = \frac{1}{l}(1 - e\cos\theta) = u \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The second part of (2) gives $f_T = 0, ie,$for elliptic orbit

$$r^2\frac{\mathrm{d}\theta}{\mathrm{dt}} = \mathrm{h} = \text{constant} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Because the force P acting on the planet/satellite is only gravitational and towards the centre S of force. With P as the force per unit mass, ie, acceleration, the first part of (2) becomes

$$f_r = \frac{d^2r}{dt^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{dt}}\right)^2 = -P \ \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

With the help of (1),(3) and (4) we need to prove that P is inversely proportional to the square of the distance r or in other words to prove the 'Inverse –square law of force 'whereas Patter's relationship[4] serves as a statement only and needless to mention that r, θ, l, e, h, t are defined[1]. Eliminating t between (2)and (4) and replacing r by $\frac{1}{u}$ one gets $(r, \theta) - \text{second} - \text{order differential equation}$

$$\frac{d}{d\theta}\left[\frac{d}{d\theta}\left(\frac{1}{u}\right)\frac{d\theta}{dt}\right]\frac{d\theta}{dt} - \frac{1}{r^3}\left(r^2\frac{d\theta}{dt}\right)^2 = -P$$

Or, $\frac{d}{d\theta}\left(-\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}\right)\frac{d\theta}{dt} - u^3h^2 = -P$

$\frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)hu^2 - u^3h^2 = -P$ ……………….. (6)

Or, $\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}$

The magnitude of P in the light of the present criterion is finally determined by use of (3) and (6):

$\frac{du}{d\theta} = \frac{e\sin\theta}{l}$

$\frac{d^2u}{d\theta^2} + u = \frac{1}{l}$ (By use of (3))

Comparing this equation with (6) we get

$P = \frac{\mu}{r^2}$ ……………………… (7)

Where $\mu = \frac{h^2}{l}$

Which espouses that a body executes an elliptical orbit under Inverse-square law of gravitation,as depicted in Figure-1.

3. Proof of Kepler's third law on assumption of his second law and Inverse –square law of gravitation

The area of the ellipse swept out by the orbiting body is

$A = \pi ab$ ………………….. (8)

Where its semi-major axis=a, semi-minor axis=b, eccentricity e and sem-latus rectum l are given by

$e^2 = \frac{a^2-b^2}{a^2}$, $l = \frac{b^2}{a}$ e<1 ……………….. (9)

Let the line joining the planet or satellite to the centre S of the force make angleθ and $\theta + \delta\theta$ in time t and t$+ \delta t$ with the initial line varying its length from r to
r+ δr. Then the area traced out in small interval δt of time Type equation here.

$dA = \frac{1}{2}r(r + \delta r)\sin\delta\theta$ (vide Figure-2)

$= 1/2\, r(r + \delta r)\delta\theta$ because $\sin\delta\theta = \delta\theta, \quad \delta\theta \to 0, \quad \delta t \to 0, \delta r \to 0$

Or, $\frac{dA}{dt} = Lim_{\delta r \to 0, \delta\theta \to 0}\frac{\delta A}{\delta\theta} = Lim_{\delta\theta \to 0}\frac{r^2}{2}\frac{\delta\theta}{\delta t} = \frac{r^2}{2}\frac{d\theta}{dt}$ ……………………. (10)

Recollecting Kepler's second law in conformity with (4),

Because of (7),(9),(11) and (8), the time period T of orbit is given by

$T = \frac{\pi ab}{h/2} = \frac{2\pi ab}{\sqrt{\mu l}} = \frac{2\pi ab}{\sqrt{\mu\frac{b^2}{a}}} = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$

$Or, T^2 = 4\pi^2a^3 \quad or, T^2 is\ proportional\ to\ a^3$…………………. (12)

Which states Kepler's second law.

4.Evaluation of Elements of Elliptic orbit subject to the Initial conditions $Projection$

An elliptic orbit of a body is specified by two elements out of a,b,e,l which depend on the initial conditions of projection.If an artificial satellite is projected from one end A distance r_0 from the focus S (centre of the Earth) of the major axis with velocity v_0 at right angles to it at time t_0 ,then in view of (4),one gets

$\left(r^2\frac{d\theta}{dt}\right)_{t=0} = r_0\left(r_0\frac{d\theta}{dt}\right)_{t=0} = r_0v_0 = h$ ……………….. (13)

And also due to (3),the initial line being the major axis,

$\frac{1}{r_0} = \frac{1-e\cos 0}{l}$ (Use of(9)) or, $\frac{1}{r_0} = \frac{1-e}{l} = \frac{1-e}{a(1-e^2)} = \frac{1}{a(1+e)}$

or, $r_0 = a(1+e)$ …………………….. (14)

Combining (7) and (13),and thereafter(9)and (14),one gets

$$l = \frac{r_0^2 v_0^2}{\mu},\quad e = 1 - \frac{r_0}{\mu} v_0^2,\quad a = \frac{\mu r_0}{2\mu - v_0^2 r_0},\ b = (r_0 v_0)\sqrt{\frac{r_0}{2\mu - v_0^2 r_0}} \text{ ……………… (15)}$$

5.Determination of time period of elliptic orbit by use of Inverse-square law of force rather than Kepler's third law

Eliminating r between (3) and (4) is obtained

$$\frac{d\theta}{dt} = \frac{h(1-e\cos\theta)^2}{l^2} \text{ ……………………. (16)}$$

And hence the time period of the orbit is given by

$$T = \int_0^{2\pi} \frac{l^2}{h(1-e\cos\theta)^2}\, d\theta = \frac{2l^2}{h}\int_0^{\pi} \frac{d\theta}{(1-e\cos\theta)^2} = \frac{2l^2}{h} I \text{ ……………….. (17)}$$

$$\text{Where } I = \int_0^{\pi} \frac{d\theta}{(1-e\cos\theta)^2} \text{ …………………. (18)}$$

Whose evaluation is an uphill task,available in many textbooks of Integral Calculus/Mechanics. Nonetheless it is done here in a shorter form by use of some properties of definite integral.

$$\int_0^{\pi} \frac{d\theta}{(1-e\cos\theta)^2} = \int_0^{\pi} \frac{d\theta}{(\{1-e\cos(\pi-\theta)\}^2} = \int_0^{\pi} \frac{d\theta}{(1+e\cos\theta)^2} \text{ …………………. (19)}$$

Adding (18) and (19) we have

$$2I = 2\int_0^{\pi} \frac{(1+e^2\cos^2\theta)d\theta}{(1-e^2\cos^2\theta)^2}$$

$$I = \int_0^{\pi} \frac{1-e^2\cos^2\theta+2e^2\cos^2\theta}{(1-e^2\cos^2\theta)^2} d\theta$$

$$= \int_0^{\pi} \frac{d\theta}{1-e^2\cos^2\theta} + 2e^2\int_0^{\pi} \frac{\cos^2\theta d\theta}{(1-e^2\cos^2\theta)^2} \quad \text{(Replacing } \cos\theta \text{ by } 1/\sec\theta)$$

$$= \int_0^{\pi} \frac{\sec^2\theta d\theta}{\sec^2\theta - e^2} + 2e^2\int_0^{\pi} \frac{\sec^2\theta d\theta}{(\sec^2\theta-e^2)^2}$$

$$= 2\int_0^{\pi/2} \frac{d(\tan\theta)}{\tan^2\theta+1-e^2} + 4e^2\int_0^{\pi/2} \frac{d(\tan\theta)}{(\tan^2\theta+1-e^2)^2}$$

$$= 2(I_1 + 2e^2 I_2) \text{ ……………………… (20)}$$

$$\text{where } I_1 = \int_0^{\pi/2} \frac{d(\tan\theta)}{\tan^2\theta+1-e^2} = \frac{\pi}{2\sqrt{1-e^2}} \text{ ……………….. (21)}$$

$$I_2 = \int_0^{\pi/2} \frac{d(\tan\theta)}{(\tan^2\theta+1-e^2)^2}$$

$$= \int_0^{\pi/2} \frac{(\tan^2\theta+1-e^2-\tan^2\theta)d(\tan\theta)}{(1-e^2)(\tan^2\theta+1-e^2)^2}$$

$$= \int_0^{\frac{\pi}{2}} \frac{d(\tan\theta)}{(1-e^2)(\tan^2\theta+1-e^2)} - \int_0^{\frac{\pi}{2}} \frac{(\tan\theta)(\tan\theta)d(\tan\theta)}{(1-e^2)(\tan^2\theta+1-e^2)^2} \quad \text{(Integrating by parts w.r.t. } \tan\theta)$$

$$= \frac{\pi}{2(1-e^2)^{\frac{3}{2}}} - \frac{1}{2(1-e^2)}\left[\lim_{\theta\to\frac{\pi}{2}} \frac{-\tan\theta}{\tan^2\theta+1-e^2} + \int_0^{\frac{\pi}{2}} \frac{d(\tan\theta)}{\tan^2\theta+1-e^2}\right] \quad \text{(vide (21))}$$

$$\text{0r, } I_2 = \frac{\pi}{4(1-e^2)^{\frac{3}{2}}} \quad \text{(value of this limit=0) ……………………. (22)}$$

Substituting (21) and (22) in relation (20), we get

$$I = \frac{\pi}{\sqrt{1-e^2}} + \frac{e^2\pi}{(1-e^2)^{\frac{3}{2}}} = \frac{\pi}{(1-e^2)^{\frac{3}{2}}} \text{ ………………… (23)}$$

Employing integral (23) in relationship (17) and applying (9),we get

$$T = \frac{2l^2}{h}.\frac{\pi}{(1-e^2)^{\frac{3}{2}}} = \frac{2l^2}{\sqrt{\mu l}}\frac{\pi}{(1-e^2)^{\frac{3}{2}}} = \frac{2\pi}{\sqrt{\mu}}\left(\frac{l}{\sqrt{1-e^2}}\right)^{3/2} = \frac{2\pi}{\sqrt{\mu}} a^{3/2} \text{ ……………….. (24)}$$

Which establishes Kepler's third law that the square of the time period is proportional to the cube of the major axis.

6.Application of Coulomb's law of attraction and repulsion in Statical Electricity:

Inverse Square law Of Charges

Example1. Let us consider two insulated strings 1and 2 of lengths c and d respectively. One end of string1 is attached to a small heavy spherical ball1 carrying positive charge q_1 and the other end of this string is fixed to a nail on a wall. Similarly one end of string2 is attached to another small heavy spherical ball 2 carrying positive charge q_2, its other end being fixed to the same point/nail. The two balls, not in touch with the wall, are found to be at rest in the same horizontal line with distance a apart. Find the weights of the balls and inclinations of the strings to the horizontal.

Solution to Example1.As is in Figure3, let B and C be two like charged small spherical balls of weights W_1 and W_2 suspended by two insulated strings BA and CA respectively,A being the nail.Let T_1 andT_2 be the tensions of the strings BA and CA inclined at angles$< B\ and < C$ to the horizontal respectively.By Coulomb's law Balls B and C are acted on by two equal and opposite repulsive forces,given by

$F=\in \frac{q_1 q_2}{a^2}$ ……………….. (25)

Where $\in$ is the permissivity of the mendium. The ball B is in equilibrium under the action of three forces:

1) Its weight W_1 acting vertically downwards,2)tensionT_1 of string1along BA and 3)force F of repulsion along CB.

For equilibrium of ball B, resolving the forces horizontally and vertically we obtain

$T_1\cos B = F$, $T_1 \sin B = W_1$ ………………………… (26)

Combining two equations of (26) and similarly considering equilibrium of ball C one gets

$\tan B= \frac{W_1}{F}$, ………………………. (27)

$\tan C=\frac{W_2}{F}$ …………………………… (28)

Using (25) in (27) and (28) we get

$W_1 = \in \frac{q_1 q_2}{a^2} \tan B$ ………………… (29)

$W_2 = \in \frac{q_1 q_2}{a^2} \tan C$ ………………………. (30)

We see that sides of triangle ABC are a,b,c with angles A,B,C, each of which with the knowledge of Trigonometry can be expressed in terms of the former in the following way:

Area of triangle ABC$=\frac{ac\sin B}{2}=\sqrt{s(s-a)(s-b)(s-c)}$ ………………… (31)

With $\cos B = \frac{c^2+a^2-b^2}{2ca}$, $2s=a+b+c$ which lead to

$\tan B=\frac{\sqrt{(a+b+c)(a-b+c)(a+b-c)(b+c-a)}}{c^2+a^2-b^2}$ ………………………. (32)

Example2. One end each of two insulated strings of length a and c are attached to either of two small heavy spherical balls carrying unlike charges q_1 and q_2 respectively. Their other ends are fixed to two nails respectively, separated by a distance a in the same horizontal line. Find the weights of the balls and inclinations of the strings to the horizontal, if they lie in the same horizontal line at rest a distance b between them.

Solution to example2. Let α and β be the inclinations of the strings BA and CD to the horizontal respectively,B and C being the positions of the balls at rest such that AB=a,BC=b,CD=c,DA=d. BC is parallel to AD. This is shown in Figure-2.

Ball B is in equilibrium, being acted on by the forces :

(1) Weight W_1 of the ball acting vertically downwards,
(2) Tension T_1 of string BA
(3) Force F of attraction by Coulomb's law given by

$F=\in \frac{q_1 q_2}{b^2}$ ($\in$= permissivity of the medium) ……………….. (33)

Resolving the forces horizontally and vertically for equilibrium of the ball B,

$T_1\cos\alpha = F$, $T_1\sin\alpha = W_1$ ………………… (34)

Which in consequence of (33) give rise to

$W_1 = \in \frac{q_1 q_2}{b^2} \tan\alpha$ ……………………. (35)

Similarly for equilibrium of the ball C

$W_2 = \in \frac{q_1 q_2}{b^2} \tan\beta$……………….. (36)

Now we need to find tanα and tanβ in terms of the sides of the trapezium $ABCD$. By geometry in Figure-2,

$a\sin\alpha - c\sin\beta = 0$ ………………….. (37)

$a\cos\alpha + c\cos\beta + d = b$

or, $a\cos\alpha + c\cos\beta = (b-d)$ ………………… (38)

Squaring and adding (37) and (38), we have on simplification

$\cos((\alpha+\beta) = \frac{\{(b-d)^2 - (a^2+c^2)\}}{2ac} = f(say)$ ………………… (39)

Eliminating a between (37) and (38),we get

$\frac{\sin(\alpha+\beta)}{\sin\alpha} = \frac{b-d}{c} = h(say)$ …………………. (40)

Eliminating c between (37) and (38), we get

$\frac{\sin(\alpha+\beta)}{\sin\beta} = \frac{b-d}{a} = k(say)$ …………………. (41)

Eliminating $(\alpha+\beta)$ $between$ (39)and (40) and between (39) and (41) by adopting 'square and adding',

$h^2 sin^2\alpha + f^2$=1

$h^2 sin^2\beta + f^2$=1

Which lead to determination of inclination α and β :

$\sin\alpha = \frac{\sqrt{1-f^2}}{h}$ ………………. (42)

$\sin\beta = \frac{\sqrt{1-f^2}}{k}$ ……………….. (43)

From which tanα and tanβ can be substituted in(35)and (36) respectively to compute W_1 and W_2 in terms of the given parameters.

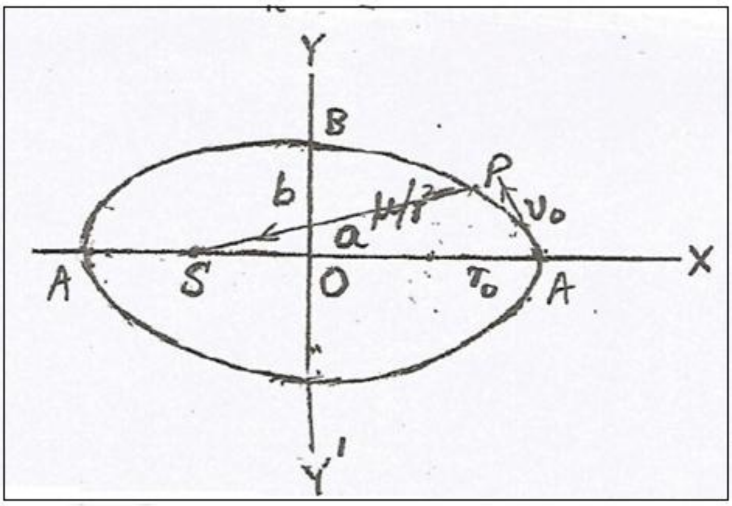

Fig-1: Elliptic Orbit of a Satellite P under Invers-Squre law of force

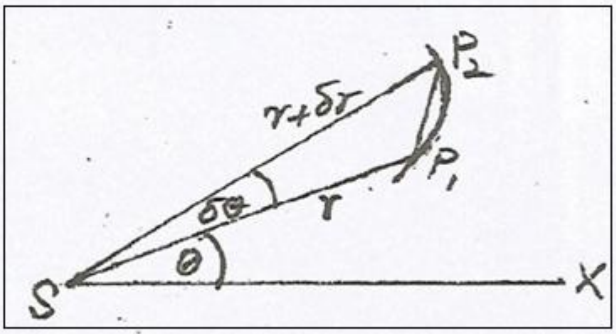

Fig-2: Area Swept out by the radius vector in small interval of time

CONCLUSION

Several centuries ago Kepler discovered laws of planetary motion by virtue of his observation, experience, vision and acumen when the Universal law of gravitation vis-à-vis the Inverse-square law of gravitation between the Sun and a planet was unknown .It is felt that the present write-up is worth classroom-teaching because of mathematical treatise with some new insight and attributes Kepler’s laws to the gravitational force between two massive objects. Attention is also drawn to an application[2] of Inverse-square law of gravitation.

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