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The Problems of Fixed Point in Mathematical Analysis

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Abstract **Review Article**

It is discussed that the problems of fixed point in mathematical analysis, and some applications of them is given as well.

Keywords: fixed point, continuous function, functional equation, limit of a sequence.

INTRODUCTION

In mathematical analysis, we will encounter problems such as proving that certain function equations have real roots in a specified interval, or determining that certain recursive sequences have limits. Although these problems belong to different types, they can all be reduced to the problem of fixed points of functions in mathematical analysis. This paper discusses the fixed points of functions within the scope of mathematical analysis, and gives some specific applications of them.

Definition 1 Let A and B be non-empty sets of numbers, $f : A \to B$ is a function defined on A, if $x^* \in A$ exists, let $f(x^*) = x^*$, Then x^* is called a fixed point of function f on A.

Let $f(A) = \{f(x) \mid x \in A\}$ be the range of the function f.

Theorem 1 The following three conditions are sufficient conditions for the fixed point of function f in its domain:

1) f is a continuous function on R, and $f(R)$ is a bounded set;
2) (a,b) is a finite interval, f is continuous on (a,b) and $f((a,b)) = R$;
3) f is continuous on $[a,b]$ and $f([a,b]) \subset [a,b]$.

Proof ⑴ Let $F(x) = f(x) - x$, then $f(R)$ be the bounded set, knowing that the general upper and lower bounds of $F(x)$ on R are $\sup_{x\in R} F(x) = +\infty$ and $\inf_{x\in R} F(x) = -\infty$. Since $F(x)$ is continuous, there are $F(R) = (-\infty, +\infty)$ according to the intermediary value, and because of $x = 0 \in F(R)$, so $x_0 \in R$ makes $F(x_0) = 0$, this is $f(x_0) = x_0$ and x_0 is the fixed point of f at R.

⑵ It can be proved that the conclusion holds imitating ⑴ .

⑶ It is known that for any $x \in [a,b]$, there is $a \le f(x) \le b$. Let $F(x) = x - f(x)$, if there is $F(a) = 0$ or $F(b) = 0$, then a or b is the fixed point of f on $[a,b]$; If $F(a) \ne 0$ and $F(b) \ne 0$, then $F(x)$ is continuous on $[a,b]$, and $F(a) = f(a) - a > 0$, $F(b) = f(b) - b < 0$, by the intermediate value theorem of the continuous function, it makes $F(x_0) = 0$ existing $x_0 \in (a,b)$ and x_0 is the fixed point of f at $[a,b]$. By using theorem 1, it can be judged that function equations such as $\sin x = x$, $(x \in R)$; $\tan x = x$, $(x \in (-\frac{\pi}{2}, \frac{\pi}{2}))$ and $e^{-x} = x$, $(x \in [0,1])$ have real roots in a given interval.

Theorem 2 Let f be a function defined on $[a,b]$, and $f([a,b]) \subset [a,b]$, if there is a constant $0 < q \leq 1$, for any x 、 $y \in [a,b]$ and $x \neq y$, having $|f(x) - f(y)| < q|x - y|$, then f has a unique fixed point on $[a,b]$.

Proof (1) When $0 < q < 1$, f is the compression map from $[a, b]$ to $[a, b]$. According to the principle of compression mapping, f has a unique fixed point x^* on $[a, b]$, and x^* is the limit of Cauchy sequence

$x_n = f(x_{n-1})$ $(x_0 \in [a,b], n = 1,2,3,\cdots)$, i.e. $x^* = \lim_{n\to\infty} x_n$.

(2) When $q = 1$, for any $x, y \in [a,b]$, $x \neq y$ having

$|f(x) - f(y)| < |x - y|$ ………………. (1)

We prove that f has a single fixed point x^* on $[a,b]$. To do this, we construct function

$$F(x) = |f(x) - x| \quad (x \in [a,b])$$

It is easy to know from the known conditions that $f(x)$ is continuous on $[a,b]$, so that $F(x)$ is also continuous on $[a,b]$, According to (1), when $f(x) \neq x$, there is

$F(f(x)) = |f(f(x)) - f(x)| < |f(x) - x| = F(x)$ ……………….. (2)

Take any $x_0 \in [a,b]$, let $x_n = f(x_{n-1})$ $(n = 1,2,3,\cdots)$, then $\{x_n\} \subset [a,b]$ is a bounded sequence. Let $\{x_{n_k}\}$ be a sub-sequence, making $\lim_{k\to\infty} x_{n_k} = \xi$. Since $[a,b]$ is a closed set, so $\xi \in [a,b]$. By the continuity of $F(x)$, having

$\lim_{k\to\infty} F(x_{n_k}) = F(\xi)$ ………………. (3)

And $F(x_{n_k+1}) = F(f(x_{n_k}))$, by the continuity of composite functions, obtaining

$\lim_{k\to\infty} F(x_{n_k+1}) = F(f(\xi))$ ………………. (4)

According to formula (2), for any natural number n, having

$F(x_{n+1}) = F(f(x_n)) < F(x_n)$,

So the sequence C is monotonically decreasing; and $F(x_n) \geq 0$ $(n = 1,2,3\cdots)$, so we know $\{F(x_n)\}$ convergence from monotone bounded principle. And $\{F(x_{n_k})\}$ and $\{F(x_{n_k+1})\}$ are both sub-sequence of $\{F(x_n)\}$, so they both converge to the same limit. So from (3) and (4), having

$F(f(\xi)) = F(\xi)$.

By formula (2), there must be $f(\xi) = \xi$. If $\eta \in [a,b]$, making $f(\eta) = \eta$, when $\eta \neq \xi$, having

$|\eta - \xi| = |f(\eta) - f(\xi)| < |\eta - \xi|$,

But this is a contradiction. So $\eta = \xi$ is the only fixed point of f on $[a,b]$, and the theorem is complete.

Inference 1For any $x, y \in [a,b]$, when $x \neq y$, there is

$|f(x) - f(y)| < |x - y|$.

Then the recursive sequence $x_n = f(x_{n-1})$ $(x_0 \in [a,b], n = 1,2,3,\cdots)$ converges to the unique solution of the equation $f(x) = x$ on $[a,b]$.

Proof In the proof of case (2) of Theorem 2, knowing by the uniqueness of the fixed point ξ of f, there must be $\overline{\lim_{n\to\infty}} x_n = \underline{\lim_{n\to\infty}} x_n = \xi$. So the sequence $\{x_n\}$ converges to the unique solution of the equation $f(x) = x$ on $[a,b]$.

In the limit of a series of mathematical analysis, the series given by the recursive formula $x_n = f(x_{n-1})$ $(x_0 \in [a,b], n = 1,2,3,\cdots)$ is often encountered. This type of sequence is often a monotonic bounded sequence, and the monotonic bound principle can be applied to determine the existence of their limits. But in fact, verifying the monotonicity of such a sequence is not always an easy task, and sometimes it is not a monotonic sequence, but with limits. At this time, we can study whether $f(x)$ is differentiable on $[a,b]$ and whether there is $|f'(x)| < 1$ on differentiable. Because under this condition, according to the differential median theorem, for any $x, y \in [a,b]$, when $x \neq y$, having

$$|f(x) - f(y)| = |f'(c)||x - y| < |x - y|$$ (Where c is between h and k).

Knowing that f also satisfies the conditions of Theorem 2, so $\{x_n\}$ must converge to the unique solution of equation $f(x) = x$ on $[a,b]$.

Example 1 For any $x_0 \in [-1,1]$, the sequence $x_n = \sin x_{n-1}$ $(n = 1,2\cdots)$ converges to the unique solution $x = 0$ of the equation $x = \sin x$ on $[-1,1]$. In fact, suppose that $f(x) = \sin x$, then $f([-1,1]) \subset [-1,1]$, and $|f'(x)| = |\cos x| < 1$, so from the inference of Theorem 2, the above conclusion holds. It should be noted that, in case (1) of Theorem 2, the domain $[a,b]$ of the function f can be changed to $I = R$ or $I = [a,+\infty)$ or $I = (-\infty, a]$. At this time, f still has a unique fixed point on these intervals. It can be shown that the sequence $x_n = f(x_{n-1})$ $(x_0 \in [a,b], n = 1,2,3,\cdots)$ still converges to the unique solution of $x = f(x)$ on I.

Example 2 Let $a > 0$, $x_1 \geq \sqrt{a}$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ $(n = 1,2,3\cdots)$ prove that $\{x_n\}$ converges, and find $\lim_{n\to\infty} x_n$.

Proof Let $f(x) = \frac{1}{2}(x + \frac{a}{x})$ $(x \geq \sqrt{a})$, it is easy to know $f([\sqrt{a},+\infty)) \subset (\sqrt{a},+\infty)$ and

$$|f'(x)| = \frac{1}{2}\left|1 - \frac{a}{x^2}\right| \leq \frac{1}{2} < 1 \quad (x \in [\sqrt{a},+\infty)),$$

So $\{x_n\}$ converges to the only fixed point x_0 of $f(x)$ in $[\sqrt{a},+\infty)$, and solve for $x_0 = a$ by $\frac{1}{2}(x_0 + \frac{a}{x_0}) = x_0$, that is $\lim_{n\to\infty} x_n = a$.

In case (2) of Theorem 2, if we change to any kind of infinite interval, its conclusion may no longer hold.

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