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Vertex Distinguishing General-total Coloring of $K_{2,5,p}$

ZHANG Shuang*, CHEN Xiang'en

College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China

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***Corresponding author:** ZHANG Shuang

Abstract **Review Article**

With the wide application of graph coloring in real life,it has gradually become one of the important fields studied by many scholars. A general total coloring of graph G refers to mapping f : $V(G)\cup E(G)\rightarrow[1,k]$.For any $x\in V(G)$,let $C_f(x)$ or $C(x)$ be the set of colors of vertex x and edges incident with x under f, which is called the color set of point x under f. For any $u,v\in V(G)$, if $C(u)\neq C(v)$, then f is called a k-vertex distinguishing general-total coloring of graph G (k-GVDTC).The minimum number of colors required for a VDIETC of G is denoted by $\chi_{gvt}(G)$, which is called the vertex distinguishing general-total chromatic number of graph G. Vertex distinguishing general-total coloring of complete tripartite graph $K_{2,5,p}$ is discussed in this paper by using the methods of distributing the color sets in advance, constructing the colorings and contradiction. The vertex distinguishing general-total chromatic numbers of $K_{2,5,p}$ are determined.

Keywords: complete tripartite graphs; general-total coloring; vertex distinguishing general-total coloring; vertex distinguishing general-total chromatic number.

AMS Subject Classification (2010):05C15

INTRODUCTION

The point distinguishing general edge coloring of graphs are raised by Harary F in [1] and then be studied in [1-6] deeply. For a total coloring (proper or not) f of G and a vertex x of G, let $C(x)$ be the set of colors of vertex x and edges incident with x under f. For a proper total coloring,if $C(u)\neq C(v)$,for any two distinct vertices u and v, then the coloring is called a vertex distinguishing (proper) total coloring, or a VDT coloring of G for short. The minimum number of colors required for a VDT coloring of G is denoted by $\chi_{gvt}(G)$.The vertex distinguishing (proper) total coloring of graph was introduced and studied in [7]. In the following we consider not necessarily proper general total coloring which are vertex distinguishing. A general total coloring f of G is an assignment of some colors to the vertices and edges of G, for any $u,v\in V(G)$, $u\neq v$,we have $C(u)\neq C(v)$,then f is called a vertex distinguishing general total coloring,or a GVDTC briefly. Vertex distinguishing general-total coloring was presented in [8].The minimum number of colors required for a GVDTC of graph G is denoted by $\chi_{gvt}(G)$.

In this paper,we consider vertex distinguishing general coloring of $K_{2,5,p}$,its general vertex distinguishing chormatic of $K_{2,5,p}$ will be determined as well. Let $X=\{x_1,x_2,\ldots,x_n\}$,$Y=\{y_1,y_2,\ldots,y_n\}$,$Z=\{z_1,z_2,\ldots,z_p\}$.$V(K_{m,n,p})=X\cup Y\cup Z$,and $E(K_{m,n,p})=\{x_iy_j\,|i=1,2,\ldots,m,j=1,2,\ldots,n\}\cup\{y_jz_t\,|j=1,2,\ldots,n,t=1,2,\ldots,p\}\cup\{x_iz_t\,|i=1,2,\ldots,m,t=1,2,\ldots,p\}$.

For convenience of description,we make the following agreements:When an l-GVDTC of a graph is mentioned or is to be given herein,we always think that the l color used is $1,2,\ldots,l$;An i-subset of $\{1,2,\ldots,l\}$ is a subset of $\{1,2,\ldots,l\}$ containing i elements;If A is a subset of $\{1,2,\ldots,l\}$,then $\overline{A}$ is used to denoted the complement set of A.

Preliminaries

Lemma 1: If $k \geq 11$ and $p > \sum_{i=1}^{8}\binom{k-1}{i}$ -6, then $K_{2,5,p}$ has no $(k$-1)-GVDTC.

Proof: Suppose $K_{2,5,p}$ has a $(k$-1)-GVDTC coloring f.

Claim 1: Any 1-subset of $\{1,2,\ldots,k\text{-}1\}$ cannot be the color sets of vertices in $X\bigcup Y$. Otherwise, if {1} is the color sets of the vertices in X,then the subsets of all vertices in Z contain color "1".The number of the subsets of $\{1,2,\ldots,k\text{-}1\}$ which may become the color sets of vertices in Z is $\binom{k-2}{1}+\binom{k-2}{2}+\cdots+\binom{k-2}{7}$.However, $p>\sum_{i=1}^{8}\binom{k-1}{i}$ -6.This is a contradiction.

Claim 2: There are at least three 1-subsets of $\{1,2,\ldots,k\text{-}1\}$,which cannot be the color sets of vertices in Z.

Otherwise, withoutloss of generality, we may assume that $\{3\},\{4\},\ldots,\{k\text{-}1\}$ are all the color sets of vertices in Z.That is to say,$C(x_i)\bigcap C(y_j)\supseteq\{3,4,\ldots,k\text{-}1\}$.Thus,the color sets of vertices in $X\bigcup Y$ can only be $\{3,4,\ldots,k\text{-}1\},\{1,3,4,\ldots,k\text{-}1\},\{2,3,4,\ldots,k\text{-}1\}$or $\{1,2,\ldots,k\text{-}1\}$,which cannot distinguish the 7 vertices in $X\bigcup Y$ .This is a contradiction.

From Claim 2,we can assume that {1},{2} or {3} cannot be the color sets of vertices in Z. Combining Claim 1,we know {1},{2} or {3} are not available for any vertex in the graph.Because $\overline{C(x_i)}$ and $\overline{C(y_j)}$,i=1,2, j=1,2,3,4,5,the 7 subsets are different,and at least 3 of them are not ∅,{1},{2} or {3}.We can assume that $\overline{C(y_3)}$, $\overline{C(y_4)}$ or $\overline{C(y_5)}$ are all not ∅,{1},{2},{3}.

(i)When $|\overline{C(y_j)}|\leq 3$,$j$=3,4,5,we know that {1},{2},{3}, $\overline{C(y_3)}$, $\overline{C(y_4)}$ and $\overline{C(y_5)}$ can not be the color sets of vertices in Z.Then the available color sets of vertices in Z are the 1-subset,2-subset,…,8-subset of $\{1,2,\ldots,k\text{-}1\}$,except for {1},{2},{3}, $\overline{C(y_3)}$, $\overline{C(y_4)}$ and $\overline{C(y_5)}$ .So $p\leq\sum_{i=1}^{8}\binom{k-1}{i}$ -6,a contradiction.

(ii)$|\overline{C(y_j)}|\geq 4$ for $j\in\{3,4,5\}$.we know that 1-subset,2-subset,3-subset and 4-subset of $\overline{C(y_j)}$ cannot be the color sets of vertices in Z. So $p\leq\sum_{i=1}^{8}\binom{k-1}{i}$ -15,a contradiction.

Lemma 2: If $k \geq 11$ and $\sum_{i=1}^{8}\binom{k-1}{i}$ -6<$p\leq\sum_{i=1}^{8}\binom{k}{i}$ -6,then $K_{2,5,p}$ exist a k-GVDTC.

Proof: In the first,we distribute subsets of $\{1,2,\ldots,k\}$ to the vertices of $K_{2,5,p}$. Put $D(x_1)=\{1,2,\ldots,k\}$, $D(x_2)=D(x_1)\backslash\{2\}$, $D(y_1)=D(x_1)\backslash\{1\}$, $D(y_2)=D(x_1)\backslash\{3\}$, $D(y_3)$ $=D(x_1)$ \ {4}, $D(y_4)$= $D(x_1)\backslash\{5\}$, $D(y_5)$= $D(x_1)\backslash\{1,3\}$, $D(z_i)=\{i+5\}$, i=1,2,…,k-5, $D(z_{k\text{-}4})=\{1,6\}$, $D(z_{k\text{-}3})=\{2,6\}$, $D(z_{k\text{-}2})=\{3,6\}$, $D(z_{k\text{-}1})=\{4,6\}$, $D(z_k)=\{5,6\}$.Let $\Re$ be the sequence,which is consist of the subsets of $\{1,2,\ldots,k\}$ with cardinal numbers between 2 and 8,except for {1,3},{1,6},{2,6},{3,6},{4,6} and {5,6},noticing that $|\Re|=\binom{k}{2}+\binom{k}{3}+\cdots+\binom{k}{8}$ -6.Let $D(z_{k+1}),D(z_{k+2}),\ldots,D(z_p)$ be the 1,2,…,p-k terms in $\Re$,respectively.This is reasonable,for $p\text{-}k\leq\binom{k}{2}+\binom{k}{3}+\cdots+\binom{k}{8}$ -6, i.e. $p\leq\sum_{i=1}^{8}\binom{k}{i}$ -6.

In the following,we give a k-GVDTC coloring f of $K_{2,5,p}$ using colors 1,2,…,k.Put $g(x_i)$=1,i=1,2,$g(y_j)$=1,j=1,2,…,5,$g(z_t)=maxD(z_t)$, t=1,2,…,p.$\forall x\in X,y\in Y$, let $g(xy)$=min$[D(x)\bigcap D(y)]$.When $u\in X\bigcup Y$,$|D(z_t)|$=2,let $g(uz_t)=min[D(u)\bigcap D(z_t)]$;When $|D(z_t)|$=3,we assume that $D(z_t)=\{a,b,c\}$ and $a<b<c$.Color edges x_1z_t, y_1z_t and x_2z_t with a,b and c.When $u\in\{y_2, y_3, y_4, y_5\}$,$g(uz_t)=min[D(u)\bigcap D(z_t)]$; When $|D(z_t)|$=4, we assume that $D(z_t)=\{a,b,c,d\}$ and $a<b<c<d$.Color edges x_1z_t , y_1z_t , x_2z_t and y_2z_t with a,b,c and d.When $u\in\{y_3,y_4,y_5\}$,$g(uz_t)=min[D(u)\bigcap D(z_t)]$;When $|D(z_t)|$=5, we assume that $D(z_t)=\{a,b,c,d,e\}$ and $a<b<c<d<e$.Color edges x_1z_t , y_1z_t , x_2z_t , y_2z_t and y_3z_t with a,b,c ,d and e.When $u\in\{y_4, y_5\}$,$g(uz_t)=min[D(u)\bigcap D(z_t)]$; When $|D(z_t)|$=6, we assume that $D(z_t)=\{a_1,a_2,a_3,a_4,a_5,a_6\}$ and $a_1<a_2<a_3<a_4<a_5<a_6$.Color edges x_1z_t , y_1z_t , x_2z_t , y_2z_t , y_3z_t and y_4z_t with a_1, a_2, a_3, a_4, a_5 and a_6,let $g(y_5z_t)=min[D(y_5)\bigcap$

$D(z_t)$]; When $|D(z_t)|$=7 or 8, we assume that $D(z_t)=\{a_j|j=1,2,\ldots,l\}$($l$=7 or 8) and $a_1<a_2<a_3<a_4<a_5<a_6<a_7(<a_8)$. Color edges x_1z_t , y_1z_t , x_2z_t , y_2z_t , y_3z_t , y_4z_t and y_5z_t with a_1, a_2, a_3, a_4, a_5 , a_6 and a_7.

It is not hard to see that $C(v)=D(v),\forall v\in V(K_{2,5,p})$.Therefore our coloring g is a vertex distinguishing general-total coloring.

MAIN RESULTS AND ITS PROOFS

Theroem 1: For any positive integer $p\geq5$, we have:

$$\chi_{gvt}(K_{2,5,p})=\begin{cases}5, & \text{when } 5\leq p\leq 18;\\ 6, & \text{when } 19\leq p\leq 50;\\ 7, & \text{when } 51\leq p\leq 114;\\ 8, & \text{when } 115\leq p\leq 242;\\ 9, & \text{when } 243\leq p\leq 498;\\ 10, & \text{when } 499\leq p\leq 1006;\\ k, & \text{when } \sum_{i=1}^{8}\binom{k-1}{i}-6<p\leq\sum_{i=1}^{8}\binom{k}{i}-6, k\geq 11\end{cases}$$

Proof: From Lemma 1 and Lemma 2, we know that if $l\geq11$ and $\sum_{i=1}^{8}\binom{k-1}{i}$-6<$p\leq\sum_{i=1}^{8}\binom{k}{i}$-6,then the conclusion is true.Now we consider the other 6 cases.

Case 1: If $499\leq p\leq1006$, then $\chi_{gvt}(K_{2,5,p})$=10.

Assume that $K_{2,5,p}$ has a 9-GVDTCg.From Claim 1 in Lemma 1,when l=10,we know that any 1-subset of {1,2,…,9} cannot be the color sets of vertices in $X\bigcup Y$ .From Claim 2 in Lemma 1,when l=10,we know that at least three 1-subsets of {1,2,…,9} cannot be any color set of vertex in Z.So we can assume that {1},{2} or {3} cannot be the color sets of vertices in the graph. Because $\overline{C(x_i)}$ and $\overline{C(y_j)}$, i=1,2, j=1,2,3,4,5,the 7 subsets are different,and at least 3 of them are not ∅,{1},{2} or {3}.We can assume that $\overline{C(y_3)}$, $\overline{C(y_4)}$ or $\overline{C(y_5)}$ are all not ∅,{1},{2},{3}.

(i)When $|\overline{C(y_j)}|\leq3$,j=3,4,5, the available color sets of vertices in Z are the 1-subset,2-subset,…,8-subset of {1,2,…,9},except for {1},{2},{3}, $\overline{C(y_3)}$, $\overline{C(y_4)}$ and $\overline{C(y_5)}$ and at least 6 color sets of vertices in $X\bigcup Y$ .So $p\leq\sum_{i=1}^{8}\binom{9}{i}$-12=498,a contradiction.

(ii)$|\overline{C(y_j)}|\leq4$ for $j\in\{3,4,5\}$.we know that 1-subset,2-subset,3-subset and 4-subset of $\overline{C(y_j)}$ cannot be the color set of any vertex in Z. So $p\leq\sum_{i=1}^{8}\binom{9}{i}$-15=496,a contradiction.

A 10-GVDTC of $K_{2,5,p}$ can be obtained by 10-GVDTC of $K_{2,5,1006}$,which is limited by $\{x_1,x_2,y_1,y_2,\ldots,y_5,z_1,z_2,\ldots,z_p\}$.Then we give a 10-GVDTC of $K_{2,5,1006}$. Put $D(x_1)=\{1,2,\ldots,10\}$, $D(x_2)=D(x_1)\backslash\{2\}$, $D(y_1)=D(x_1)\backslash\{1\}$, $D(y_2)=D(x_1)\backslash\{3\}$, $D(y_3)=D(x_1)\backslash\{4\}$,$D(y_4)=D(x_1)\backslash\{5\}$, $D(y_5)=D(x_1)\backslash\{1,3\}$. Put 1-subset, 2-subset, …, 8-subset of {1,2,…,10},except for {1},{2},{3},{4},{5},{1,3},as the color sets of vertices in Z.By the second paragraph of Lemma 2,we could get the specific coloring method of $K_{2,5,1006}$.

Case 2: If $243\leq p\leq498$,then $\chi_{gvt}(K_{2,5,p})$=9.

Assume that $K_{2,5,p}$ has a 8-GVDTCg.From Claim 1 in Lemma 1,when l=9,we know that any 1-subset of {1,2,…,8} cannot be the color set of vertices in $X\bigcup Y$ .From Claim 2 in Lemma 1,when l=9,we know that at least three 1-subset of {1,2,…,8} cannot be a color set of vertex in Z.So we can assume that {1},{2} or {3} are not the color sets of vertices in the graph. Because $\overline{C(x_i)}$ and $\overline{C(y_j)}$,i=1,2, j=1,2,3,4, 5,the 7 subsets are different,and at least 3 of them are not $\varnothing$,{1},{2} or {3}.We can assume that $\overline{C(y_3)}$, $\overline{C(y_4)}$ or $\overline{C(y_5)}$ are all not $\varnothing$,{1},{2},{3}.

(i)When $|\overline{C(y_j)}|\le 3$,j=3,4,5, the available color set of vertices in Z are the 1-subset,2-subset,…,8-subset of {1,2,…,8},except for {1},{2},{3}, $\overline{C(y_3)}$, $\overline{C(y_4)}$ and $\overline{C(y_5)}$ and all color sets of vertices in $X\bigcup Y$ .So $p\le\sum_{i=1}^{8}\binom{8}{i}$-13=242,a contradiction.

(ii)$|\overline{C(y_j)}|\le 4$ for $j\in\{3,4,5\}$.we know that 1-subset,2-subset,3-subset and 4-subset of $\overline{C(y_j)}$ cannot be the color set of any vertex in Z. So $p\le\sum_{i=1}^{8}\binom{8}{i}$-15=240, a contradiction.

A 9-GVDTC of $K_{2,5,p}$ can be obtained by 9-GVDTC of $K_{2,5,498}$,which is limited by $\{x_1,x_2,y_1,y_2,\ldots,y_5,z_1,z_2,\ldots,z_p\}$.Then we give a 9-GVDTC of $K_{2,5,498}$.Put $D(x_1)$={1,2,…,9}, $D(x_2)=D(x_1)\backslash\{2\}$, $D(y_1)= D(x_1)\backslash\{1\}$, $D(y_2)= D(x_1)\backslash\{3\}$, $D(y_3)= D(x_1)\backslash\{4\}$,$D(y_4)= D(x_1)\backslash\{5\}$, $D(y_5)= D(x_1)\backslash\{1,3\}$. Put 1-subset, 2-subset, …, 8-subset of {1,2,…,9},except for {1},{2},{3},{4},{5},{1,3},as the color sets of vertices in Z.By the second paragraph of Lemma 2,we could get the specific coloring method of $K_{2,5,498}$.

Case 3: If 115≤*p*≤242, then χ_{gvt} ($K_{2,5,p}$)=8.

Assume that $K_{2,5,p}$ has a 7-GVDTCg.From Claim 1 and Claim 2 in Lemma 1,when l=8,we can assume that {1},{2} or {3} are not the color sets of vertices in the graph. Because $\overline{C(x_i)}$ and $\overline{C(y_j)}$,i=1,2, j=1,2,3,4,5,the 7 subsets are different,and at least 3 of them are not $\varnothing$,{1},{2} or {3}.We can assume that $\overline{C(y_3)}$, $\overline{C(y_4)}$ or $\overline{C(y_5)}$ are all not $\varnothing$,{1},{2},{3}.

(i)When $|\overline{C(y_j)}|\le 3$,j=3,4,5, the available color set of vertices in Z are the 1-subset,2-subset,…,7-subset of {1,2,…,7},except for {1},{2},{3}, $\overline{C(y_3)}$, $\overline{C(y_4)}$ and $\overline{C(y_5)}$ and all color sets of vertices in $X\bigcup Y$ .So $p\le\sum_{i=1}^{7}\binom{7}{i}$-13=114,a contradiction.

(ii)$|\overline{C(y_j)}|\le 4$ for $j\in\{3,4,5\}$.we know that 1-subset,2-subset,3-subset and 4-subset of $\overline{C(y_j)}$ can not be the color set of any vertex in Z. So $p\le\sum_{i=1}^{7}\binom{7}{i}$-15=112,a contradiction.

A 8-GVDTC of $K_{2,5,p}$ can be obtained by 8-GVDTC of $K_{2,5,242}$,which is limited by $\{x_1,x_2,y_1,y_2,\ldots,y_5,z_1,z_2,\ldots,z_p\}$.Then we give a 8-GVDTC of $K_{2,5,242}$.Put $D(x_1)$={1,2,…,8}, $D(x_2)=D(x_1)\backslash\{2\}$, $D(y_1)= D(x_1)\backslash\{1\}$, $D(y_2)= D(x_1)\backslash\{3\}$, $D(y_3)= D(x_1)\backslash\{4\}$,$D(y_4)= D(x_1)\backslash\{5\}$, $D(y_5)= D(x_1)\backslash\{1,3\}$. Put 1-subset, 2-subset, …, 8-subset of {1,2,…,8},except for {1},{2},{3},{4},{5},{1,3},as the color sets of vertices in Z.By the second paragraph of Lemma 2,we could get the specific coloring method of $K_{2,5,242}$.

Case 4: If 51≤*p*≤114, then χ_{gvt} ($K_{2,5,p}$)=7.

Assume that $K_{2,5,p}$ has a 6-GVDTCg.From Claim 1 and Claim 2 in Lemma 1,when l=7,we can assume that {1},{2} or {3} are not the color sets of vertices in the graph. Because $\overline{C(x_i)}$ and $\overline{C(y_j)}$,i=1,2, j=1,2,3,4,5,the 7 subsets are different, and at least 3 of them are not $\varnothing$,{1},{2} or {3}.We can assume that $\overline{C(y_3)}$, $\overline{C(y_4)}$ or $\overline{C(y_5)}$ are all not $\varnothing$,{1},{2},{3}.

(i)When $|\overline{C(y_j)}|\le 3$,j=3,4,5, the available color sets of vertices in Z are the 1-subset,2-subset,…,6-subset of {1,2,…,6},except for {1},{2},{3}, $\overline{C(y_3)}$, $\overline{C(y_4)}$ and $\overline{C(y_5)}$ and all color sets of vertices in $X\bigcup Y$ .So $p\le\sum_{i=1}^{6}\binom{6}{i}$-13=50,a contradiction.

(ii)$|\overline{C(y_j)}|\leq 4$ for $j\in\{3,4,5\}$.we know that 1-subset,2-subset,3-subset and 4-subset of $\overline{C(y_j)}$ cannot be the color set of any vertex in Z. So $p\leq\sum_{i=1}^{6}\binom{6}{i}$ -15=48,a contradiction.

A 7-GVDTC of $K_{2,5,p}$ can be obtained by 7-GVDTC of $K_{2,5,114}$,which is limited by $\{x_1,x_2,y_1,y_2,\ldots,y_5,z_1,z_2,\ldots,z_p\}$.Then we give a 7-GVDTC of $K_{2,5,114}$. Put $D(x_1)=\{1,2,\ldots,7\}$, $D(x_2)=D(x_1)\backslash\{2\}$, $D(y_1)=D(x_1)\backslash\{1\}$, $D(y_2)= D(x_1)\backslash\{3\}$, $D(y_3)= D(x_1)\backslash$

$\{4\}$, $D(y_4)=D(x_1)\backslash\{5\}$, $D(y_5)=D(x_1)\backslash\{1,3\}$. Put 1-subset, 2-subset, …, 7-subset of {1,2,…,7},except for {1},{2},{3},{4},{5},{1,3},as the color sets of vertices in Z. By the second paragraph of Lemma 2,we could get the specific coloring method of $K_{2,5,114}$.

Case 5: If $19\leq p\leq 50$,then $\chi_{gvt}(K_{2,5,p})=6$.
Assume that $K_{2,5,p}$ has a 5-GVDTCg.From Claim 1 and Claim 2 in Lemma 1,when l=6,we can assume that {1},{2} or {3} are not the color sets of vertices in the graph. Because $\overline{C(x_i)}$ and $\overline{C(y_j)}$,i=1,2, j=1,2,3,4,5,the 7 subsets are different, and at least 3 of them are not ∅,{1},{2} or {3}.We can assume that $\overline{C(y_3)}$, $\overline{C(y_4)}$ or $\overline{C(y_5)}$ are all not ∅,{1},{2},{3}.

(i)When $|\overline{C(y_j)}|\leq 3$,$j$=3,4,5, the available color sets of vertices in Z are the 1-subset,2-subset,…,5-subset of {1,2,…,5},except for {1},{2},{3}, $\overline{C(y_3)}$, $\overline{C(y_4)}$ and $\overline{C(y_5)}$ and all color sets of vertices in $X\bigcup Y$ .So $p\leq\sum_{i=1}^{5}\binom{5}{i}$ -13=18,a contradiction.

(ii)$|\overline{C(y_j)}|\leq 4$ for $j\in\{3,4,5\}$.we know that 1-subset,2-subset,3-subset and 4-subset of $\overline{C(y_j)}$ cannot be the color set of any vertex in Z. So $p\leq\sum_{i=1}^{5}\binom{5}{i}$ -15=16,a contradiction.

A 6-GVDTC of $K_{2,5,p}$ can be obtained by 6-GVDTC of $K_{2,5,50}$,which is limited by $\{x_1,x_2,y_1,y_2,\ldots,y_5,z_1,z_2,\ldots,z_p\}$.Then we give a 6-GVDTC of $K_{2,5,50}$. Put $D(x_1)=\{1,2,\ldots,6\}$, $D(x_2)=D(x_1)\backslash\{2\}$, $D(y_1)=D(x_1)\backslash\{1\}$, $D(y_2)=D(x_1)\backslash\{3\}$, $D(y_3)=D(x_1)\backslash\{4\}$,$D(y_4)=D(x_1)\backslash\{5\}$, $D(y_5)=D(x_1)\backslash\{1,3\}$. Put 1-subset, 2-subset, …, 6-subset of {1,2,…,6},except for {1},{2},{3},{4},{5},{1,3},as the color sets of vertices in Z. By the second paragraph of Lemma 2,we could get the specific coloring method of $K_{2,5,50}$.

Case 6: If $5\leq p\leq 18$,then $\chi_{gvt}(K_{2,5,p})=5$.
Assume that $K_{2,5,p}$ has a 5-GVDTCg.

Claim 3: Any 1-subset of {1,2,3,4} cannot be the color sets of vertices in X.

Otherwise, we can assume that $g(x_1)=\{1\}$,then any vertex in $Y\bigcup Z$ contains color "1".There are only 8 subsets which contain color "1",so they cannot distinguishing at least 11 vertices in $Y\bigcup Z\bigcup\{x_1\}$,a contradiction.

Claim 4: There are at least three 1-subsets of {1,2,3,4},which cannot be the color sets of vertices in Z.

Otherwise, we can assume that only {1} or {2} are not the color sets of vertices in Z.$C(x_i)\bigcap C(y_j)\supseteq\{3,4\}$,so the color sets which can be assigned are:{3,4},{1,3,4},{2,3,4} and {1,2,3,4}.That 4 sets can not distinguish the 7 vertices in $X\bigcup Y$ .This is a contradiction.

Claim 5: When $p\geq 6$,any 1-subset of {1,2,3,4} can not be the color sets of vertices in Y.

Otherwise, we could assume $g(y_1)=1$,then any vertex in $X\bigcup Z$ must contain color "1".There are only 8 subsets which contain color "1",so they can not distinguish at least 9 vertices in $X\bigcup Z\bigcup\{y_1\}$,a contradiction.

The following are discussed in two cases:
(i)When $p\geq 6$,

From Claim 2,we can assume {1},{2},{3} can not be the color sets of vertices in Z. Because $\overline{C(x_i)}$ and $\overline{C(y_j)}$,i=1,2, j=1,2,3,4,5,the 7 subsets are different,and at least 3 of them are not ∅,{1},{2} or {3}.We can assume that $\overline{C(y_3)}$, $\overline{C(y_4)}$ or $\overline{C(y_5)}$ are all not ∅,{1},{2},{3}.Then the available color sets of vertices in Z are the 1-subset,2-subset,3-subset,4-subset of {1,2,3,4},except for {1},{2},{3}, $\overline{C(y_3)}$, $\overline{C(y_4)}$, $\overline{C(y_5)}$ and all color sets of vertices in $X \bigcup Y$ .So $p \le \sum_{i=1}^{4}\binom{4}{i}$-13=2,a contradiction.

(ii)When p=5,

{1,2,3,4} has 15 subsets,they could be the color sets of vertices in $K_{2,5,5}$.There are 14 subsets except for {1,2,3,4},which are complementary to each other.So there are 7 pairs of sets which are complementary to each other.The color sets in X and Y,Y and Z,Z and X must not complement each other.We consider if there are two vertices complementing to each other in X,then any 1-subset cannot be the color sets of vertices in Z. $p+7 \le \sum_{i=1}^{4}\binom{4}{i}$-4, $p \le 4$,a contradiction. Similarly, the color sets of any two vertices in Y or Z cannot complement each other.So there are only 7 subsets which could be the color sets of vertices in $K_{2,5,5}$, a contradiction.

A 5-GVDTC of $K_{2,5,p}$ can be obtained by 5-GVDTC of $K_{2,5,18}$,which is limited by $\{x_1,x_2,y_1,y_2,\ldots,y_5,z_1,z_2,\ldots,z_p\}$.Then we give a 5-GVDTC of $K_{2,5,18}$. Put $D(x_1)=\{1,2,\ldots,5\}$, $D(x_2)=D(x_1)\backslash\{2\}$, $D(y_1)= D(x_1)\backslash\{1\}$, $D(y_2)= D(x_1)\backslash\{3\}$, $D(y_3)= D(x_1)\backslash\{4\}$,$D(y_4)= D(x_1)\backslash\{5\}$, $D(y_5)= D(x_1)\backslash\{1,3\}$. Put 1-subset, 2-subset, …, 5-subset of {1,2,…,5},except for {1},{2},{3},{4},{5},{1,3},as the color sets of vertices in Z.By the second paragraph of Lemma 2,we could get the specific coloring method of $K_{2,5,18}$.

REFERENCES

1. Harary F, Plantholt M. The point -distinguishing chromatic index [M]:F. HARARY, MAYBEE (Eds.) J.S., Graphs and Application, Wiley interscience, New York. 1985:147-162.
2. Horvnak M, Sotak R. The fifth jump of the point-distinguishing chromatic index of $K_{n,n}$ [J]. Ars Combinatoria. 1996, 42:233-242.
3. Horňák M, Soták R. Localization of jumps of the point-distinguishing chromatic index of $ K_ {n, n} $. Discussiones Mathematicae Graph Theory. 1997;17(2):243-51.
4. ZAGAGLIA SALVI N. On the point -distinguishing chromatic index of $K_{n,n}$. Ars Combinatoria. 1988, 25B: 93-104.
5. Zagaglia Salvi N. On the value of the point-distinguishing chromatic index of $K_{n,n}$ [J]. Ars Combinatoria, 1990, 29B: 235-244.
6. Horvnak M, Zagaglia Salvi N. On the point-distinguishing chromatic index of $K_{m,n}$ [J]. Ars Combinatoria, 2006, 80: 75-85.
7. Zhang Z, Qiu P, Xu B, Li J, Chen X, Yao B. Vertex-distinguishing total coloring of graphs. Ars Combinatoria. 2008 Apr 1;87(2):33-45.
8. Liu C, Zhu E. General vertex-distinguishing total coloring of graphs. Journal of Applied Mathematics. 2014;2014.