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Attractors for the Nonclassical Diffusion Equations with Fading Memory and White Noise

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Abstract **Review Article**

In this paper, the longtime behavior of nonclassical diffusion equations with memory decay and white noise is considered. It is proved that the initial value problem has a solution and a random attractor.

Keywords: Nonclassical diffusion equation, Random attractor, White noise.

INTRODUCTION

In this paper, we study the long time dynamical behavior of solutions for the following nonclassical diffusion equations with fading memory and white noise:

$$\begin{cases} u_t - \alpha\Delta u_t - \Delta u - \int_0^\infty k(s)\Delta\mu(t-s)ds = f(u) + \sum_{j=1}^m h_j d\omega_j, \quad in\,\Omega, \\ u(x,t) = 0, on\,\partial\Omega, \\ u(x,t) = u_0(x,t), x\in\Omega, t\le 0. \end{cases} \tag{1.1}$$

While Ω is a bounded domain in $R^n(n\ge 3)$. with respect to the stochastic term $\sum_{j=1}^m h_j d\omega_j$,

We do the $O-U$ change for the stochastic term. For the nonlinearity, we presume that f is a Lipschitz continuous function and satisfies:

$$\lim_{|s|\to\infty}\sup\frac{f(s)}{s} < \lambda_1. \tag{1.2}$$

$$|f'(s)| \le C(1+|s|^{\frac{4}{n-2}}) \ , \ s\in R. \tag{1.3}$$

$$|f'(s)| \le C(1+|s|^p). \forall s\in R. \begin{cases} p\le \frac{4}{n-4}, n\ge 5, \\ p\ge 0. \end{cases} \tag{1.4}$$

Where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. and we assume $f(0)=0$.

He effects of fading memory in this equation are shown through the linear time convolution of the function $\Delta u(\cdot)$ and the memory kernel $k(\cdot)$. We assume $k(\cdot)\in C^2(R^+), k(s)\le 0$, and $k'(s)\le 0, \forall s\in R^+$. Beside, we also assume that the function $\mu(s) = -k'(s)$ and satisfies

$$\mu \in C^1(R^+) \cap L^1(R^+),\ \mu(s) \geq 0,\ \mu'(s) \leq 0,\ \forall s \in R^+, \tag{1.5}$$

$$\mu'(s) + \delta\mu(s) \leq 0,\ \forall s \geq 0. \tag{1.6}$$

Where δ is a positive constant? Obviously, it follows the exponential decay of the kernel $k(s)$ and $\mu(s)$ zero. This behavior shows the fading memory of the distant past in the model we will be talking about.

In 1980. Aifantis in [1] point out the classical reaction-diffusion equation

$$u_t - \Delta u = f(u) + g \tag{1.7}$$

It does not cover every aspect of reaction-diffusion problems. It ignores the viscosity, elasticity and pressure of the intermediates in the solid diffusion process. In addition, diffuse solid Aifantis was also found. For example, the energy structure equation is different when the conducting medium is under pressure. Viscoelastic (memory loss). He built up a mathematical model with some concrete examples.It contains the viscosity, elasticity and pressure of the medium, namely, the non-classical diffusion equation:

$$u_t - \Delta u_t - \Delta u = f(u) + g \tag{1.8}$$

In 2010, Xuan Wang in [13] consider viscoelasticity of the conductive medium, that is, the author add fading memory term to Eq(1.8), the equation is the nonclassical diffusion equation with fading memory that is the equation:

$$u_t - \Delta u_t - \Delta u - \int_0^\infty k(s)\Delta\mu(t-s)ds = f(u) + g(x) \tag{1.9}$$

The speed of energy dissipation for Eq (1.9) is faster than for usual nonclassical equation. The conduction of energy is not only affected by present external forces but also by historic external forces.

This equation appear as extension of usual nonclassical diffusion equation in fluid mechanics, solid mechanics and heat conduction theory (see [1, 5, 6]).

Since Eq (1.9) contains the term $-\Delta u_t$, it is different from the usual reaction diffusion equations with fading memory almost, usually has a fading memory of reaction diffusion equations with high regularity, but for the equation (1.9), if the initial data only belongs to the weak topological space, then the solution is always in the weak topology space, no higher regularity, because of $-\Delta u_t$, therefore, for autonomy, cannot use compact Sobolev embedding to verify the key to the solution of semigroup asymptotic compactness.

The long-time behavior of the solution of Eq (1.8) has been studied for the autonomous case in [8, 9, 11, 12]. In Y.Xiao [12]. The author has proved the existence of global attractors in $H_0^1(\Omega)$, when nonlinearity is subcritical and $g(x) \in L^2(\Omega)$, In C.Sun, M. Yang [8]. The author have testified the existence of global attractor when nonlinearity is critical and $g(x) \in H^{-1}(\Omega)$. and then Xuan Wang in [23] prove the existence of global attractors in the weak topological space $H_0^1(\Omega) \times L_\mu^2(R^+; H_0^1(\Omega))$ and the strong topological space $D(\mathrm{A}) \times L_\mu^2(R^+; D(\mathrm{A}))$.

Now, if we consider adding a random term of white noise to Eq (1.9), then the equation is a non-classical diffusion equation with memory decay and white noise that we will study. Eq (1.1) has not been considered before, and this paper first studies it as a new model to prove the existence of random attractors.Since Eq(1.1) contains memory terms, we first construct a relatively complex solution space, and do norm inner product in this space. The $O-U$ transformation is applied to the random term.

Attractor is an important concept in the study of asymptotic behavior of deterministic dynamical systems. Crauel, Debussche and Flandoli [15] proposed a general theory for the study of random attractors by defining the attractor set as the attractor of any set of orbitals starting from minus infinity. The random attractor is a compact invariant set, which is dependent on chance and moves with time. The existence of random attractors for two-dimensional random Navier-Stokes equations is proved by using the theory. In this paper, another method is used to prove the existence of random attractors for the long time behavior of nonclassical diffusion equations with memory decay in white noise. By using the method of operator decomposition (see [19]), the asymptotic compactness of the solution of the system (1.1) is established, which is a key step to obtain the random attractor.

The arrangement of this article is as follows. The second part gives the relevant concepts and theories. In Section 3, we introduce the Ornstein-Ohlenbeck procedure and some properties, and provide some basic Settings for (1.1). In section 4, we prove the existence of a unique random attractor for a stochastic dynamical system generated by (1.1).

Notation and preliminaries

In this section, we recall some basic notions of the theory of random dynamical system (RDS) (see [14, 15, 17, 20, 21]) and Kuratowski of non-compactness (see [18]), which is a useful tool to study the attractor (see [19, 22]).

Let $(\mathrm{X}, \|\cdot\|_{\mathrm{X}})$ be a separable Banach space with Borel $\sigma-a\lg ebra$ $\mathrm{B}(\mathrm{X})$ and $(\Omega, F, P(\vartheta_t)_{t\in R})$ be the ergodic metric dynamical system.

Defintion 2.1 [23] A continous random dynamical system over $(\Omega, F, P(\vartheta_t)_{t\in R})$ is a $(B(R^+))\times F\times B(\mathrm{X})$, $B(X)-$measurable maping.

$$S: R^+\times\Omega\times X\rightarrow X(t,\omega,x)\rightarrow S(t,\omega,x).$$

Satisfying the following properties

(1) $S(0,\omega,x)=x$ for $\omega\in\Omega$ and $x\in X$;

(2) $S(t+\tau,\omega,\cdot)=S(t,\vartheta_\tau\omega,\cdot)\circ S(\tau,\omega,\cdot)$ for τ , $\omega\geq 0$,and $\omega\in\Omega$;

(3) S is continous with respect to x for $t\geq 0$ and $\omega\in\Omega$.

A set-vauled map $B:\Omega\rightarrow 2^X$ is called a random closed set if $B(\omega)$ is a nonempty closed set and $\omega\rightarrow d(x,B(\omega))$ is measurable for $x\in X$. A random set $B(\omega)$ is called tempered if for P-$a.s.$ $\omega\in\Omega$ and all $\beta>0$

$$\lim_{t\to\infty} e^{-\beta t}\sup\{\|b\|_X : b\in B(\vartheta_{-t}\omega)\}=0.$$

Let D be the collection of all tempered random subsets in X and $\{K(\omega)\}_{\omega\in\Omega}\in D$. Then $\{K(\omega)\}_{\omega\in\Omega}$ is called a random absorbing set for S in D if for $B(\omega)\in D$ and P-$a.e.$ $\omega\in\Omega$, there exist $t_B(\omega)>0$ such that

$$S(t,\vartheta_{-t}\omega, B(\vartheta_{-t}\omega))\subset K(\omega) \text{ for all } t\geq t_B(\omega).$$

Definition 2.2 [23] A random set $\{\mathrm{A}(\omega)\}\in D$ is random attractor (or pullback attractor) for a RDS S if the following conditions are satisfied, for P-$a.s.$ $\omega\in\Omega$,

(i) $A(\omega)$ is a random compact set. $i.e.\omega\rightarrow d(x,A(\omega))$ is measurable for every $x\in X$ and $A(\omega)$ is compact;

(ii) $\{A(\omega)\}$ is strictly invariant, $i,e.$

$$S(t,\omega,A(\omega))=A(\vartheta_t\omega) \text{ for all } t\geq 0;$$

(iii) $\{A(\omega)\}$ attracts every set in D , $i.$ $e.$ for all $B=\{B(\omega)\}\in D$,

$$\lim_{t\to\infty} d_H(\varphi(t,\vartheta_{-t}(\omega)), B(\vartheta_{-t}\omega), A(\omega))=0;$$

Where d_H is the Hausdorff semi-distance.

Let B be a bounded set is a Banach space X .The Kuratowski measure of non-compactness $\alpha(B)$ of B is defined by

$$\alpha(B)=\inf\{ d>0: B \text{ admits a finite cover by sets of diameter} \leq d \}.$$

We define $\alpha(B)=\infty$, if B is unbounded, see [8].

Definition 2.3. [19] A random dynamical system S on a Polish space (X,d) is almost surely $D-\alpha-contracting$ if

$$\lim_{t\to\infty}\alpha(S(t,\vartheta_{-t}\omega, A(\vartheta_{-t}\omega)))=0 \text{ for } A\in D.$$

Lemma 2.4. For a random dynamical system $S(t,\omega)$ on a separabal Banach space $(X, \|\cdot\|_X)$, if almost surely the following hold:

(1) $S(t,\omega) = S_1(t,\omega) + S_2(t,\omega)$;

(2) For any tempered random variable $a \geq 0$, there exist $r(a)$ $(0 \leq r \leq \infty)$, $a.s.$ such that for the closed ball B_a with radius a in X, $S_1(t, \vartheta_{-t}\omega, B_a(\vartheta_{-t}\omega))$ is precompact *in* X for all $t > r(a)$.

(3) $\|S_2(t, \vartheta_{-t}\omega, u)\|_X \leq K(t, \vartheta_{-t}\omega, a)$, $t > 0$, $u \in B_a(\omega)$ *and* $K(t,\omega,a)$ *is a measurable function with respect to* (t,ω,x) *which satisfies*

$$\lim_{t\to\infty} K(t, \vartheta_{-t}\omega, a) = 0$$

Then $S(t,\omega)$ *is almost surely* $D-\alpha-contracting$ (*see* [19]).

Lemma 2.5. Let $S(t,\omega)$ be a random dynamical system on a Polish space $(X, \|\cdot\|_X)$.
Assume that

(1) $S(t,\omega)$ has an absorbing set $B(\omega) \in D$;

(2) $S(t,\omega)$ is almost surely $D-\alpha-contracting$.

Then $S(t,\omega)$ possesses a global random attractor in X.

The basic setting

Now, we consider the one-dimensional Ornstein-Uhlenbeck equation

$$dz_j + \lambda z_j dt = d\omega_j(t) \tag{3.1}$$

So

$$z_j(t) = z_j(\vartheta_t\omega_j) = -\lambda\int_{-\infty}^{0} e^{\lambda\tau}(\tau)d\tau, \text{ for } t \in R$$

Putting $z(\vartheta_t\omega) = \sum_{j=1}^{m}(I-\alpha\Delta)^{-1}h_j z_j(\vartheta_t\omega_j)$, *where* Δ *is* the Laplacian with domain $H_0^1 \cap H^2(0,l)$, By (3.1) we have

$$dz - \alpha d(\Delta z) + \alpha(z - \alpha z)dt = \sum_{j=1}^{m} h_j d\omega_j \tag{3.2}$$

Lemma 3.1. [23] For $\lambda > 0$, there exist a tempered random variable $\rho_1 : \Omega \to R$ such that

$$\|z(\vartheta_t\omega)\|_{2p+2} \leq e^{\lambda|t|}\rho_1(\omega) \text{ for } t \in R \text{ and } \omega \in \Omega,$$

Where $\rho \geq 0$ and $\rho_1(\omega)$, $\omega \in \Omega$ satisfies:

$$\rho_1(\vartheta_t\omega) \leq e^{\lambda|t|}\rho_1(\omega) \text{ for } t \in R.$$

Corollary 3.2. [23] For $\lambda > 0$, there exists a tempered random variable $\rho_2 : \Omega \to R$ such that for $\hbar = 0$ or 1,

$$\left\|A^{\frac{\hbar}{2}} z(\vartheta_t\omega)\right\| + \alpha\sqrt{\lambda_1}\left\|A^{\frac{1+\hbar}{2}} z(\vartheta_t\omega)\right\| \leq e^{\lambda|t|}\rho_2(\omega).$$

For $t \in R$ and $\omega \in \Omega$, where λ_1 is the first eigenvalue of $-\Delta$.

As in [3], we introduce a new variable which reflect the past history of the Eq (1.1), that is,

$$\eta^t(x,s) = \int_0^s u(x, t-r)dr, \; s \geq 0, \tag{3.3}$$

Then

$$\partial_t \eta^t(x,s) = u(x,t) - \partial_s \eta^t(x,s),\ s \geq 0. \tag{3.4}$$

In view of assumption about memory kernel $\mu(\cdot)$, let $L^2_\mu(R^+; H_r)$ be the family of Hilbert space of function φ: $R^+ \to H_r$, $0 < r < 3$, endowed with the inner product and norm, respectively

$$\langle \varphi_1, \varphi_2 \rangle_{\mu, H_r} = \int_0^\infty \mu(s) \langle \varphi_1, \varphi_2 \rangle_{H_r} ds.$$

$$\|\varphi\|_{\mu, H_r} = \int_0^\infty \mu(s) \|\varphi(s)\|^2_{H_r} ds.$$

Now we introduce the family of Hilbert spaces

$$M_r = H_r \times L^2_\mu(R^2; H_r).$$

And endowed norm

$$\|z\|_{M_r} = \|(u, \eta^t)\|_{M_r} = \left(\frac{1}{2} \|u\|^2_{H_r} + \|\eta^t\|^2_{\mu, H_r} \right)^{\frac{1}{2}}.$$

In order to estimate conveniently, we first show the preliminary result as follows (cf. [2,4,7]).

Lemma 3.3. [13] Setting, Let memory kernel satisfy, then for any, there exist a constant, such that:

$$\langle \eta^t, \eta^t_s \rangle_{\mu, H_r} \geq \frac{\delta}{2} \|\eta^t\|^2_{\mu, H_r}. \tag{3.5}$$

We also need the following results to prove the asymptotic compactness about memory term as well as the existence of global attractors.

Lemma 3.4. (See [2,4,7]) Assuming that $\mu \in C^1(R^+) \cap L^1(R^+)$ is a nonnegative function, and satisfies: if the exist $s_0 \in R^+$, such that $\mu(s_0) = 0$, then $\mu(s) = 0$ for all $s \geq s_0$ holds.

Moreover, Let B_0, B_1, B_2 be Banach space, here B_0, B_1 are reflexive and satisfy $\qquad B_0 \to B_1 \to B_2.$

Where the embedding $B_0 \to B_1$ is compact, let $C \subset L^2_\mu(R^+; B_1)$ satisfy

(i) C in $L^2_\mu(R^+; B_0) \cap H^1_\mu(R^+; B_2)$;

(ii) $\sup_{\eta \in C} \|\eta(s)\|^2_{B_1} \leq \hbar(s), \forall s \in R^+, \hbar(s) \in L^1_\mu(R^+)$;

Then C is relatively compact in $L^2_\mu(R^+; B_1)$.

Lemma 3.5. (See [17]) Let H be a complete metric space, $\{S(t)\}_{t \geq 0}$ be a semigroup in and has a bounded absorbing set B_0 in H_0. If for every $t \geq 0$, the operator $S(t)$ allows the decomposition $S(t) = S_1(t) + S_2(t)$, and satisfies:

(i) The semigroup $\{S_1(t)\}_{t \geq 0}$ is uniformly compact,as t is increasing sufficiently;

(ii) The operator $S_2(t): H \to H$ is continuous, and for any bounded set $B \subset H$, as $t \to \infty$

$$r_B(t) = \sup_{\varphi \in B} \|S_2(t)\varphi\|_H \to 0$$

Then ω -limit set of absorbing set is global attractors of $\{S(t)\}_{t \geq 0}$.

Let $v(t,\omega)=u(t,\omega)-z(\vartheta_t\omega)$. Then setting $\mu(s)=-k'(s)$ and using assumption $k(\infty)=0$, Eq (1.1) can be transfored into the following system:

$$v_t-\alpha\Delta v_t-\Delta v-\int_0^\infty\mu(s)\Delta\eta(s)ds= f\left(v+z(\vartheta_t\omega)\right)+\left(1-\alpha^2\right)\Delta z(\vartheta_t\omega)+\alpha z(\vartheta_t\omega), \quad (3.6)$$

$$v(x,t)=0\ x\in\partial\Omega,\ t\geq 0, \quad (3.7)$$

$$v(x,0)=v_0(x),\ x\in\Omega, \quad (3.8)$$

$$\eta^t(x,s)=0,\ (x,s)\in\partial\Omega\times R^+,\ t\geq 0, \quad (3.9)$$

We set $E_0=H_0^1\times L^2(0,l)$, $E_1=H^2\cap H_0^1\times H_0^1\cap L^2(0,l)$. Then $E_1\to E_0$ with compact imbedding.

By a Galerkin method as in [10], it can be proved that under assumptions (1.4), for $P-a.e.\omega\in\Omega$ and for every $(v_0,\eta^0)\in E_0$, problem(3.6)-(3.9) have a unique solution $v\in C(R^+,E_0)$ and the solution (v) is continuous with respect to . Hence, the solution mapping generates a RDS. It is called stochastic flow associated with the nonlinear strain wave equation with additive noise.

Uniform time a priori estimates and random attractors

Let $E_0=H_0^1\times L^2(0,l)$ endowed with the inner product and norm $(Y_1,Y_2)_{E_0}=v_{H_0^1}+\eta^t_{L^2_\mu}$, $\|Y\|_{E_0}=\|v\|_{H_0^1}+\|\eta^t\|_{L^2_\mu}$, $Y_j=(v_j,\eta^t)$, $Y_0=(v_0,\eta^0)$. We define the following unifor estimates in E_0.

Lemma 4.1. Suppose that f satisfies (1.2)-(1.4). Then for $B=\{B(\omega)\}_{\omega\in\Omega}\in D$ $\ Y_0=(v_0,\eta^0)\in B(\omega)$, and for $P-a.e.\omega\in\Omega$, there exists $T=T(B,\omega)>0$, such that

$$\|Y(t,\vartheta_{-t}\omega,Y_0(\vartheta_{-t}\omega))\|_{E_0}\leq R(\omega)\ \text{ for }\ t\geq T,$$

Where $R(\omega)=c(1+\rho_1^{2p+2}(\omega)+\rho_2^2(\omega))$ is a positive random function.

Proof. Taking the inner product of (3.8) with v and using $v=u-z(\vartheta_t\omega)$, we have

$$\frac{d}{dt}\phi_0(t,\omega)+H_0(t,\omega)=0, \quad (4.1)$$

Where

$$\phi_0(t,\omega)=\frac{1}{2}(\|v\|^2+\alpha\|\nabla v\|^2+\|\eta^t\|^2_{\mu,E_0}), \quad (4.2)$$

$$H_0(t,\omega)=\|\nabla v\|^2+\langle\eta^t(s),\eta^t_s(s)\rangle_{\mu,E_0}-\langle f(v+z(\vartheta_t\omega)),v\rangle$$
$$-\langle(1-\alpha^2)\nabla z(\vartheta_t\omega),v\rangle-\langle\alpha z(\vartheta_t\omega),v\rangle. \quad (4.3)$$

In condition of (1.2)-(1.4), we have

$$\langle f(v+z(\vartheta_t\omega)),v\rangle\leq\lambda_1\|v+z(\vartheta_t\omega)\|^2-w\|v\|^2+c$$
$$\leq\frac{3(\lambda_1-w)}{2}\|v\|^2+\frac{\lambda_1-w}{2}\|z(\vartheta_t\omega)\|^2+c$$
$$\leq(1-\frac{w}{\lambda_1})\|\nabla v\|^2+\frac{\lambda_1-\omega}{3}\|z(\vartheta_t\omega)\|^2+c. \quad (4.4)$$

Here $w>0$, and

$$\langle(1-\alpha^2)\nabla z(\vartheta_t\omega),v\rangle+\langle\alpha z(\vartheta_t\omega),v\rangle$$

$$\leq \frac{w}{4\lambda_1}\|\nabla v\|^2 + \frac{\lambda_1(1-\alpha^2)}{2w}\|\nabla z(\vartheta_t\omega)\|^2 + \frac{\lambda_1\alpha}{2w}\|z(\vartheta_t\omega)\|^2. \tag{4.5}$$

Apply lemma (3.3), we know

$$\langle \eta^t(s), \eta_s^t(s)\rangle_{\mu,E_0} \geq \frac{\delta}{2}\|\eta^t\|_{\mu,E_0}^2. \tag{4.6}$$

Then, we get

$$\frac{1}{2}\frac{d}{dt}(\|v\|^2 + \alpha\|\nabla v\|^2 + \|\eta^t\|_{\mu,E_0}^2) + \frac{\sigma}{2}(\|v\|^2 + \alpha\|\nabla v\|^2 + \|\eta^t\|_{\mu,E_0}^2) \leq c_1(1+p_0(\vartheta_t\omega)).$$

Where

$$\sigma = \min\left\{\frac{w}{2}, \frac{w}{2\lambda_1\alpha}, \delta\right\}.$$

And

$$p_0(\vartheta_t\omega) = \|\nabla z(\vartheta_t\omega)\|^2 + \|z(\vartheta_t\omega)\|^2.$$

Obviously,

$$\|Y(t)\|_{E_0}^2 \leq \phi_0(t,\omega) \leq c_0\|Y(t)\|_{E_2}^2. \tag{4.7}$$

According to Gronwall lemma,

$$\|Y(t)\|_{E_0}^2 \leq c_0\|Y(0)\|_{E_0}^2 e^{-\sigma t} + \frac{c_1}{\sigma}p_0(\vartheta_t\omega). \tag{4.8}$$

Hence, by (4.1) and (4.6), we have

$$\frac{d}{dt}\phi_0(t,\omega) + \lambda\phi_0(t,\omega) \leq c_1 p_0(\vartheta_t\omega) + c_1, \tag{4.9}$$

Where $\lambda = \frac{\sigma}{2} > 0$, By lemma 3.1. with $\tilde{\lambda} = \frac{\lambda}{4p+4}$ and Corollary 3.2. with $\tilde{\lambda} = \frac{\lambda}{4}$, for $P-a.e.\ \omega \in \Omega$ and $t \in R$, we obtain:

$$p_0(\vartheta_t\omega) \leq 2e^{\frac{1}{2}\lambda|t|}(\frac{1}{2}\rho_1^{2p+2}(\omega) + \frac{1}{2}\rho_2^2(\omega)). \tag{4.10}$$

It follows from (4.9) that for all $t \geq 0$

$$\phi_0(t,\omega) \leq e^{-\lambda t}\phi_0(0,\omega) + c_1\int_0^t e^{\lambda(\tau-t)}p_0(\vartheta_\tau\omega)d\tau + \frac{c_1}{\lambda}. \tag{4.11}$$

Replacing ω by $\vartheta_{-t}\omega$ with $t \geq 0$ in (4.11) and using (4.10), we get:

$$\begin{aligned}\phi_0(t,\omega) &\leq e^{-\lambda t}\phi_0(0,\vartheta_{-t}\omega) + c_1\int_0^t e^{\lambda(\tau-t)}p_0(\vartheta_{\tau-t}\omega)d\tau + \frac{c_1}{\lambda}\\ &\leq e^{-\lambda t}\phi_0(0,\vartheta_{-t}\omega) + c_1\int_{-t}^0 e^{\lambda\tau}p_0(\vartheta_\tau\omega)d\tau + \frac{c_1}{\lambda}\\ &\leq e^{-\lambda t}\phi_0(0,\vartheta_{-t}\omega) + c_1\int_{-t}^0 e^{\frac{1}{2}\lambda\tau}(\rho_1^{2p+2}(\omega) + \rho_2^2(\omega))d\tau + \frac{c_1}{\lambda}\\ &\leq e^{-\lambda t}\phi_0(0,\vartheta_{-t}\omega) + c_2(1+\rho_1^{2p+2}(\omega) + \rho_2^2(\omega)),\end{aligned} \tag{4.12}$$

Where $c_2 = \frac{3c_1}{\lambda}$ is a deterministic positive constant. This together with (4.8) show that

$$\|Y(t,\vartheta_{-t}\omega, Y_0(\vartheta_{-t}\omega)\|_{E_0}^2 \leq c_0\|Y(\vartheta_{-t}\omega)\|_{E_0}^2 e^{-\sigma t} + c_*(1+\rho_1^{2p+2}(\omega) + \rho_2^2(\omega)), \tag{4.13}$$

Where $c_* = \dfrac{c_1(1+e^{-\lambda t})}{\sigma}$ is a deteminstic positive constant.

Denote

$$K(\omega)=\left\{Y\in E_0 : \|Y\|_{E_0}\le R(\omega)\right\}.$$

Then $\{K(\omega)\}_{\omega\in\Omega}\in D$ is an absorbing set in E_0 .

In order to prove that RDS $S(t,\omega)$ is almost surely D-α-$contracting$ on E_0 by Lemma 2.5. we decompose the solution $Y=(v,\eta^t)$ of the (3.6)-(3.9)with the initial value $Y_0=(v_0,\eta^0)$ into two part. Define by $Y(v^a,\eta^t)=S_1(t)(v_0,\eta^0)$ is the solution of the equations

$$v_t^a-\alpha\Delta v_t^a-\Delta v^a-\int_0^\infty \mu(s)\Delta\eta(s)ds= f(v^a+z(\vartheta_t\omega)). \tag{4.14}$$

With $(v^a,\eta^t)\big|_{t=0}=(v_0,\eta^0)$ and homogeneous boundary condition. Then

$$Y^b=(v^b,\eta^t)=S_2(t)(v_0,\eta^0)=S(t)(v_0,\eta^0)-S_1(t)(v_0,\eta^0),$$

Is the solution of the problems

$$v_t^b-\alpha\Delta v_t^b-\Delta v^b-\int_0^\infty \mu(s)\Delta\eta(s)ds= f(v+z(\vartheta_t\omega))+(1-\alpha^2)\Delta z(\vartheta_t\omega)+\alpha z(\vartheta_t\omega), \tag{4.15}$$

With the initial data $(v^b,\eta^t)\big|_{t=0}=(0,0)$ and homogeneous boundary conditions.

Lemma 4.2. Assume $Y_0=(v_0,\eta^0)\in B(\omega)\in D$, and (1.2)-(1.4) hold, Then

$$\left\|Y^a(t,\vartheta_{-t}\omega,Y_0(\vartheta_{-t}\omega))\right\|_{E_0}^2\le C\left\|Y_0(\vartheta_{-t}\omega)\right\|_{E_0}^2 e^{-\lambda_2 t}.$$

Proof Taking the inner product of (3.8) with v^a and using $v^a=u^a - z(\vartheta_t\omega)$, we have

$$\frac{d}{dt}\phi_1(t,\omega)+H_1(t,\omega)=0, \tag{4.16}$$

Where

$$\phi_1(t,\omega)=\frac{1}{2}(\|v^a\|^2+\alpha\|\nabla v^a\|^2+\|\eta^t\|_{\mu,E_0}^2),$$

And

$$H_1(t,\omega)=\|\nabla v^a\|^2+\langle\eta^t(s),\eta_s^t(s)\rangle_{\mu,E_0}.$$

Hence, we get

$$\frac{1}{2}\frac{d}{dt}(\|v^a\|^2+\alpha\|\nabla v^a\|^2+\|\eta^t\|_{\mu,E_0}^2)+\|\nabla v^a\|^2+\langle\eta^t(s),\eta_s^t(s)\rangle_{\mu,E_0}=0.$$

By using Lemma 3.3.and λ_1 is the first eigenvalue of $-\Delta$ we have

$$\frac{1}{2}\frac{d}{dt}(\|v^a\|^2+\alpha\|\nabla v^a\|^2+\|\eta^t\|_{\mu,E_0}^2)+\lambda_2\left(\|v^a\|^2+\alpha\|\nabla v^a\|^2+\|\eta^t\|_{\mu,E_0}\right)\le 0.$$

So, we obtain

$$\frac{d}{dt}\phi_1(t,\omega)+\lambda_2\phi_1(t,\omega)\le 0,$$

Where $\lambda_2=\min\left\{w,\dfrac{w}{\lambda_1\alpha},\delta\right\}$. Applying Gronwall's lemma, we botain for all $t\ge 0$

$$\phi_1(t,\omega)\le\phi_1(0,\omega)e^{-\lambda_2 t}.$$

By arguments similar to (4.7), we can get derive that

$$\left\|Y^a(t,\omega,Y_0(\omega))\right\|_{E_0}^2 \le c\left\|Y_0(\omega)\right\|_{E_0}^2 e^{-\lambda_2 t}. \tag{4.17}$$

Replacing ω by $\vartheta_{-t}\omega$ with $t \ge 0$ in (4.17), implies that the result hold.

Lemma 4.3. Assume $Y_0 = (v_0, \eta^0) \in B(\omega) \in D$ and (1.4) hold, Then

$$\left\|Y^b(t,\vartheta_{-t}\omega,Y_0(\vartheta_{-t}\omega))\right\|_{E_0}^2 \le 2R_*^2(\omega),$$

Where $R_*^2(\omega) = c_{**}(1+\rho_1^{2p+2}(\omega)+\rho_2^2(\omega))$ and c_{**} is a deterministic positive constant.

Proof Taking the inner product of (4.14) with v^b, we obtain:

$$\frac{d}{dt}\phi_2(t,\omega) + H_2(t,\omega) = 0, \tag{4.18}$$

Where

$$\phi_2(t,\omega) = \frac{1}{2}(\left\|v^b\right\|^2 + \alpha\left\|\nabla v^b\right\|^2 + \left\|\eta^t\right\|_{\mu,E_0}^2).$$

And

$$H_2(t,\omega) = \left\|\nabla v^b\right\|^2 + \left\langle \eta^t(s), \eta_s^t(s)\right\rangle_{\mu,E_0} - \left\langle f(v+z(\vartheta_t\omega)), v^b\right\rangle - \left\langle (1-\alpha^2)\nabla z(\vartheta_t\omega), v^b\right\rangle - \left\langle \alpha z(\vartheta_t\omega), v^b\right\rangle.$$ Note that using (1.2)-(1.4) with (4.6) and Yang inequality

$$\begin{aligned}\left| f(v+z(\vartheta_t\omega)), v^b\right| &\le (\lambda_1 - w)\left\|v+z(\vartheta_t\omega)\right\|\left\|v^b\right\| \\ &\le (\lambda_1 - w)(\frac{\|v\|^2}{2} + \left\|v^b\right\|^2 + \frac{\left\|z(\vartheta_t\omega)\right\|^2}{2}) \\ &\le \frac{\lambda_1 - w}{2}\|v\|^2 + (1-\frac{w}{\lambda_1})\left\|v^b\right\|^2 + \frac{\lambda_1 - w}{2}\left\|z(\vartheta_t\omega)\right\|^2.\end{aligned}$$

$$\left\langle (1-\alpha^2)\nabla z(\vartheta_t\omega), v^b\right\rangle + \left\langle \alpha z(\vartheta_t\omega), v^b\right\rangle \le \frac{w}{4\lambda_1}\left\|\nabla v^b\right\|^2 + \frac{\lambda_1}{w}(1-\alpha^2)\left\|\nabla z(\vartheta_t\omega)\right\|^2 + \frac{\lambda_1}{w}\alpha\left\|z(\vartheta_t\omega)\right\|^2.$$

Then, we can get

$$\frac{1}{2}(\left\|v^b\right\|^2 + \alpha\left\|\nabla v^b\right\|^2 + \left\|\eta^t\right\|_{\mu,E_0}^2) + \lambda_3(\left\|v^b\right\|^2 + \alpha\left\|\nabla v^b\right\|^2 + \left\|\eta^t\right\|_{\mu,E_0}^2) \le c_1(1+p_0(\vartheta_t\omega)) + c\|v\|^2, \tag{4.19}$$

Where

$$\lambda_3 = \min\left\{\frac{w}{2}, \frac{w}{2\lambda_1\alpha}, \frac{\delta}{2}\right\}.$$

Hence, we have

$$\frac{d}{dt}\phi_2(t,\omega) + \lambda_3\phi_2(t,\omega) \le c_3 p_0(\vartheta_t\omega) + c_4(1+\|v\|^2). \tag{4.20}$$

By Gronwall inequuality, we obtain

$$\phi_2(t,\omega) \le c_3\int_0^t e^{\lambda_3(s-t)} p_0(\vartheta_s\omega)ds + c_4\int_0^t e^{\lambda_3(s-t)}\|v(s)\|^2 ds + \frac{1}{\lambda_3}c_4. \tag{4.21}$$

Noeting $\phi_2(t,\omega) \ge \left\|Y^b(t,\omega)\right\|_{E_1}^2$ and replacing ω by $\vartheta_{-t}\omega$, we have

$$\|Y^b(t,\vartheta_{-t}\omega)\|_{E_1}^2$$
$$\le c_3\int_0^t e^{\lambda_3(s-t)}p_0(\vartheta_{s-t}\omega)ds + c_4\int_0^t e^{\lambda_3(s-t)}\|v(s,\vartheta_{-t}\omega)\|^2 ds + \frac{1}{\lambda_3}c_4 \tag{4.22}$$
$$\le c_3\int_{-t}^0 e^{\lambda_3 s}p_0(\vartheta_s\omega)ds + c_4\int_0^t e^{\lambda_3(s-t)}\|v(s,\vartheta_{-t}\omega)\|^2 ds + \frac{1}{\lambda_3}c_4.$$

By Corollary 3.2. the first term on the rirht hand side of () satixfies

$$c_3\int_{-t}^0 e^{\lambda_3 s}p_0(\vartheta_s\omega)ds \le c_3\int_{-t}^0 e^{\frac{1}{2}\lambda_3 s}\rho_2^2(\omega)ds \le \frac{2}{\lambda_3}c_6\rho_2^2(\omega) \text{ for } t\ge 0. \tag{4.23}$$

By (4.12), we can get

$$\|v(s,\vartheta_{-t}\omega)\|^2 \le c_0\|Y(\vartheta_{-t}\omega)\|_{E_0}^2 e^{-\lambda_3 s} + c_*(1+\rho_1^{2p+2}(\omega)+\rho_2^2(\omega)). \tag{4.24}$$

Thus, the second term on the right hand side of () satisfies for $t\ge 0$

$$c_4\int_0^t e^{\lambda_3(s-t)}\|v(s,\vartheta_{-t}\omega)\|^2 ds \le c_0c_4te^{-\lambda_3 t}\|Y(\vartheta_{-t}\omega)\|_{E_0}^2 + \frac{c_4}{\lambda_3}c_*(1+\rho_1^{2p+2}(\omega)+\rho_2^2(\omega)). \tag{4.25}$$

Let $R_*^2(\omega)=\frac{1}{\lambda_3}(2c_3+c_4+c_4c_*)(1+\rho_1^{2p+2}(\omega)+\rho_2^2(\omega))$. It follows from (4.22)-(4.25) that

$$\|Y^b(t,\vartheta_{-t}\omega)\|_{E_1}^2 \le c_0c_4te^{-\lambda_3 t}\|Y(\vartheta_{-t}\omega)\|_{E_0}^2 + R_*^2(\omega).$$

Since $\{B(\omega)\}_{\omega\in\Omega}\in D$ is tempered and $Y_0(\vartheta_{-t}\omega)\in B(\vartheta_{-t}\omega)\eta^t$, there exist $T_B^*(\omega)>0$ such that for $t\ge T_B^*(\omega)$.

$$c_0c_4te^{-\lambda_3 t}\|Y(\vartheta_{-t}\omega)\|_{E_0}^2 \le R_*^2(\omega).$$

Thus

$$\|Y^b(t,\vartheta_{-t}\omega,Y_0(\vartheta_{-t}\omega))\|_{E_0}^2 \le 2R_*^2(\omega), \tag{4.26}$$

And the result holds.

We are now in a position to present our main result:

- **Theorem 4.4** Assume (1.4) hold. Then the random dynamic system $S(t,\omega)$ has a unique random attractor in E_0.
- **Proof.** By Lemma 2.4, Lemma 4.2 and Lemma 4.3, the stochastic dynamical system $S(t,\omega)$ of the nonlinear is almost surely D-α-$contracting$. This together with Lemma2.5 implies that the existence of a unique $D-$ random attractor for $S(t,\omega)$.

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