Scholars Journal of Engineering and Technology
Abbreviated Key Title: Sch J Eng Tech
ISSN 2347-9523 (Print) | ISSN 2321-435X (Online)
Journal homepage: https://saspublishers.com

New Relationship Derived for Area-Mach, Area-Deflection and Area-Pressure for a Fully Expanded Convergent-Divergent Nozzle

R. C. Mehta[1]*

[1]Department of Aeronautical Engineering, Noolul Islam University, Kumaracoil 629180, India

DOI: https://doi.org/10.36347/sjet.2026.v14i05.006 | **Received:** 03.04.2026 | **Accepted:** 18.05.2026 | **Published:** 23.05.2026

***Corresponding author:** R. C. Mehta
Department of Aeronautical Engineering, Noolul Islam University, Kumaracoil 629180, India

Abstract **Original Research Article**

Analytical solutions for supersonic flow inside a convergent-divergent nozzle are derived using quasi-one-dimensional isentropic flow equation. Differential equations are obtained and summed over infinitesimal contribution under a reversible adiabatic process. The area-Mach, the area-deflection, and the pressure-deflection relations are obtained employing a fully expanded supersonic nozzle. First time a constant of integration is found for the solution of the area-Mach relation and nomogram is drawn for easy analysis of a convergent-divergent nozzle.

Keywords: Compressible flow, convergent-divergent nozzle, isentropic flow, nomogram, Mach wave, Prandtl-Meyer function.

1. INTRODUCTION

Compressible flow through a convergent-divergent conical nozzle is customarily analyzed using a quasi-one-dimensional isentropic flow equation described in a number of textbooks [1-7]. These are fundamental gas dynamics equations for the well-known Laval nozzle that was invented in 1988 by Swedish inventor de Laval. The method of characteristics [1, 3] in conjunction with the Prandtl-Meyer function and isentropic relation is employed to design a two-dimensional contour nozzle for obtaining uniform flow at the nozzle exit and minimum nozzle length. The method of characteristics is applicable to supersonic flow, and is widely applied to design contour nozzles because of its simplicity and the programming ease of the calculation procedure [3]. Analytical expressions for a supersonic flow inside a nozzle may derive employing quasi-one-dimensional isentropic relation and considering the limit of smooth flow over a centered expansion wave. The change of speed across a weak shock is related to the equation of continuity, Euler equation of motion with isentropic process in conjunction with supersonic centered expansion fan. Differential equations are obtained and integration is carried out under a reversible adiabatic process.

The main purpose of the analysis is to obtain analytical expression of area-Mach, area-deflection and pressure-deflection with the help of a quasi-one-dimensional isentropic flow equation. The paper derives some quantitative relations between the infinitesimal flow through a Mach wave and a flow direction. The change in the flow parameters through a Mach line wave will also be infinitesimal. A nomogram is constructed for a convergent-divergent nozzle.

2.0 Analysis

2.1 The Area-Velocity relation

We consider steady compressible adiabatic flow in a nozzle of varying cross-section as depicted in Fig. 1. Flow velocity is taken to be steady uniform over each cross-section area of a convergent-divergent nozzle. The flow through a convergent-divergent nozzle is shown in the figure the differential form [1] of the steady continuity equation may be written as

$$\frac{d}{dx}(\rho u A) = 0 \qquad (1)$$

The effects of compressibility are considered by irrotational and adiabatic flow in a convergent-divergent nozzle of varying area. Taking the logarithmic differential of Eq. (1) we obtain

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0 \qquad (2)$$

Continuity equation for steady adiabatic, compressible in a stream tube of varying cross-sectional area, compressibility may be formed by using Euler's equation [1, 4]. Employing the momentum equation and introducing Mach number, this gives relationship between density and velocity change can be written as

$$\frac{d\rho}{\rho} = -M^2 \frac{du}{u} \qquad (3)$$

The above equations represent relation between area-velocity employing equations of continuity in a logarithmic differential form, Euler equation of motion with isentropic process [1]. The momentum term represents Bernoulli equation of compressible flow to give relation between density and velocity change. Substituting Equation (3) in Equation (1) shows the area-velocity relation as

$$\frac{du}{u} = \frac{1}{(M^2-1)} \frac{dA}{A} \qquad (4)$$

Equation (4) shows the relation which relates change in area dA to change in velocity du. The sign of ($M^2 - 1$) is the determining factor [1] in the qualitative nature of flow.

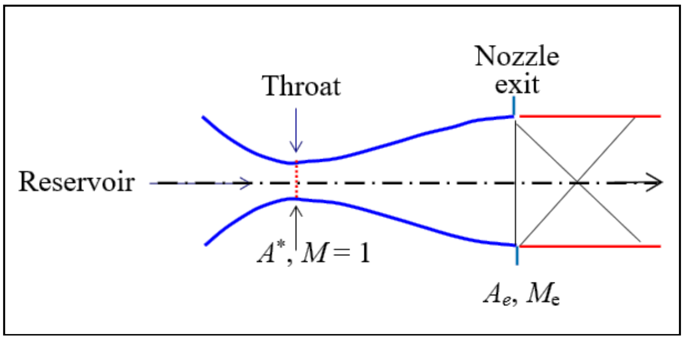
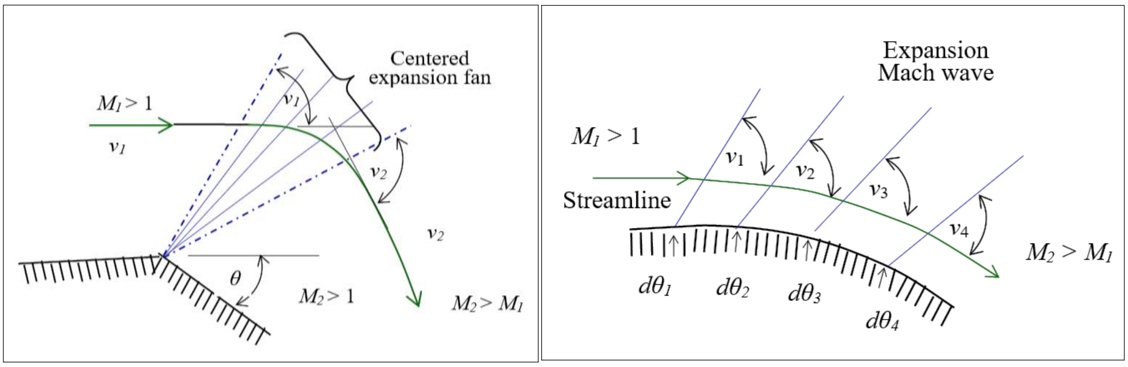


Fig. 1 Quasi-one-dimensional fully-expanded flow in a convergent-divergent nozzle

2.2 Mach wave

Now we consider the centered expansion wave at the throat. The simple expansion at a corner occurs through a centered wave, defined by a "fan" of straight line as delineated in Fig. 2 (a). The flow up to the corner is uniform, at Mach M_1, and thus the leading Mach wave must be straight line, at the Mach angle μ_1. The terminating Mach line stands at the angle μ_2 to the downstream wall. The centered wave, more often called a Prandtl-Meyer expansion fan, is the counterpart for a convex corner. We consider the limit of smooth flow [1], that is, the velocities and flow inclinations must be discontinuous but their derivatives may still be discontinuous on the Mach line.



A **B**

Fig. 2 Supersonic expansion wave (a) sharp corner body (b) smooth curve body

Any deflection of the stream due to the presence body must begin at Mach wave. The changes its direction by the amount $d\theta$ at the corner point. $d\theta$ is positive in the clockwise direction for a deflection away from flow direction. The flow follows the body surface, then the direction will also change at that point of the corner, the turning will occur at the Mach line emanating from the corner point. There is no pressure differential along the wave. The speed and Mach number in the uniform flow upstream of the wall deflection is M_1, the flow expansion may be considered as being carried out by going through a series of small amplitude directions. Figure 2 displays the Prandtl-Meyer expansion wave generated by a finite flow direction. Prandtl-Meyer flow ($M \geq 1$) is shown in the schematic diagram. The Mach line is considered as a characteristic of the flow. In isentropic flow, any flow boundary may be replaced by a streamline and the Prandtl-Meyer expansion can be considered for flow around a smooth curve or round surface as shown in Fig. 2(b).

2.3 Flow parameters with respect to Mach number

The importance of Mach number and ratio of specific heats in determining the physical properties of steady, compressible, irreversible adiabatic flow can be written

$$\frac{T_o}{T} = \left(1 + \frac{\gamma-1}{2}M^2\right) \quad (5)$$

If the flow is also reversible, the isentropic relation can be employed to express as T_o/T in terms of p_o/p and ρ_o/ρ. Thus, we get following isentropic relations

$$\frac{p_o}{p} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}} \quad (6)$$

$$\frac{\rho_o}{\rho} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}} \quad (7)$$

The variables in these equations are Mach number and ratio of specific heats. Plots of ratios of static to stagnation properties (T_o/T), (p_o/p), and (ρ_o/ρ) versus Mach number are shown in Fig. 3 for specific ratio of heats 1.4 for air. In the figure the dotted lines represent the value of flow parameters at $M = 1$.

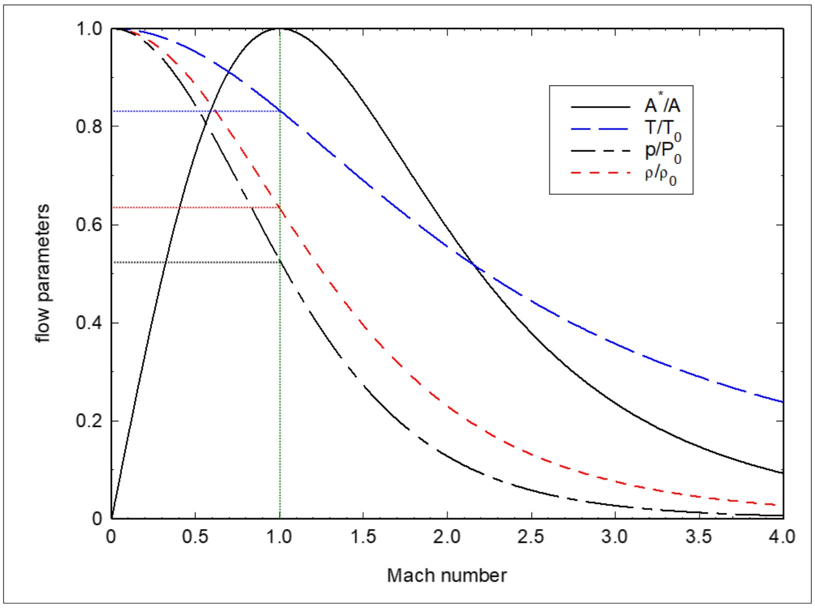


Fig. 3: Variation of flow parameters versus Mach number

2.4 The Area-Mach relation

The differential equation for an isentropic expansion by isentropic turn across weak shock or Mach wave as depicted in Fig. 2 may be written as

$$\frac{1}{V}\frac{dV}{d\theta} = -\frac{1}{\sqrt{M^2-1}} \quad (8)$$

Equation (8) is a scalar equation that relates the change in speed to a change in the velocity direction. V is the resultant velocity and θ is the inclination of the velocity vector. Equation (8) relates the infinitesimal change in velocity dV, to the infinitesimal deflection $d\theta$ across the wave of vanishing strength. It is important to mention here that the increment velocity du [1] is in limit of algebraic increment in velocity dV [4]. Equation (8) is scalar and relates change in speed to change in velocity direction. Flow properties are uniform along each Mach line. The magnitude of velocity at any point depends only on the flow direction at that point. The fluid speed is related to the local speed of sound with the adiabatic equation and the perfect gas law [1] as

$$\frac{dV}{V} = \left[\frac{1}{\left(1+\frac{\gamma-1}{2}M^2\right)}\right]\frac{dM}{M} \quad (9)$$

Equation (9) is a deflection for dV/V strictly in terms of M. Substituting Equation (4) in Equation (8) we get

$$\frac{dA}{A} = \frac{(M^2-1)}{\left(1+\frac{\gamma-1}{2}M^2\right)}\frac{dM}{M} \quad (10)$$

Equation (10) applies continuously through the isentropic turn when integrated, it gives relation between Area and Mach number, which for the present we shall simply write in the form: $A = \mathrm{F}(M)$. The exact solution of the above equation can be obtained by separating variables, and integrating. Integration of the above equation is

$$\ln A = \frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)\ln\left[1 + \frac{(\gamma-1)}{2}M^2\right] - \frac{1}{2}\ln[(\gamma-1)M^2] + c_1 \quad (11)$$

where c_1 is the constant of integration. To eliminate the c_1, let us consider two sections of the nozzle having a ratio of areas A_2/A_1. Equation (11) becomes

$$\frac{A_1}{A_2} = \frac{M_2}{M_1}\left(\frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2}\right)^{\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)} \quad (12)$$

The minimum cross-sectional area A^* for the isentropic flow occurs at $M = 1$. At $A_2 = A^*$ at $M_2 = 1$, the above equation yields

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left\{1 + \frac{\gamma-1}{2}M^2\right\}\right]^{\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)} \quad (13)$$

The ratio A/A^* is very important, it is called the area-Mach number relation. The above equation represents area-Mach relation and may also be derived using quasi-one-dimensional continuity equation in conjunction with momentum equation and isentropic relations [1, 3]. The value of A/A^* is always greater than unity, and for any given value of area ratio there is always corresponding two values of Mach. Ratios of (A/A_e) are also presented in Fig. 3 for subsonic and supersonic Mach number, respectively. The variation of (A/A_e) versus Mach number in Fig. 3 shows very interesting behavior for subsonic and supersonic Mach numbers. The (A/A_e) versus Mach number consists of subsonic and supersonic branches, so that (A/A_e) is double-valued except at $M = 1$, where $p^*/p = 0.528$ for $\gamma = 1.4$. If a throat of area A^* does not occur, the flow remains supersonic or subsonic throughout. Downstream of the throat of area A^*, the flow can either accelerate supersonically or decelerate sub-sonically, depending on the pressure ratio at the exit of the nozzle. The constant of integration c_1 is

$$c_1 = A^* - \frac{(\gamma+1)^{\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)}}{(\gamma-1)^{\frac{1}{2}}} \quad (14)$$

In Equation (13), the A/A^* relation refers as 'turned-inside out' and tells us that $M = F(A/A^*)$. Differentiation of Equation (12) can be written [8] as

$$F'(M) = M\left[\frac{2}{(\gamma+1)}\left(1 + \frac{\gamma-1}{2}M^2\right)\right]^{\frac{(3-\gamma)}{2(\gamma-1)}} - \left(\frac{A}{A^*}\right) \quad (15)$$

We can also rearrange Eq. (12) as

$$F(M) + \left(\frac{A}{A^*}\right) = \frac{1}{M}\left[\frac{2}{(\gamma+1)}\left(1 + \frac{\gamma-1}{2}M^2\right)\right]^{\frac{(\gamma+1)}{2(\gamma-1)}} \quad (16)$$

The differentiation of Eq. (16) is written [9] as

$$F'(M) = \left[\frac{2}{(\gamma+1)}\left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{(\gamma+1)}{2(\gamma-1)}} - \frac{1}{M^2} + \frac{(\gamma-1)}{2\left(1+\frac{\gamma-1}{2}M^2\right)}\right] \quad (17)$$

Comparing differentiation of Equations (12) and (16) of area-Mach relation show considerable difference. Figure 4 displays the plot of $F(M)$ and $F'(M)$. Figure 4 shows the variation of $F(M)$ in the convergent-divergent nozzle as a function of Mach number. $F'(M)$ shown in Fig. 4 is the derivative of $F(M)$ with respect to Mach number. It can be seen from Figure 4 that $F'(M)$ is nearly zero near sonic Mach number at the throat. Figure 4 highlights the convergence problem in the Newton-Raphson iteration up to compute gradient in the vicinity of $M \leq 1.2$. The geometrical parameters of the conical nozzle also influence $F'(M)$. The value of $F'(M)$ becomes very sensitive to the area ratio of convergent-divergent nozzle as can be seen in Equations (15, 17).

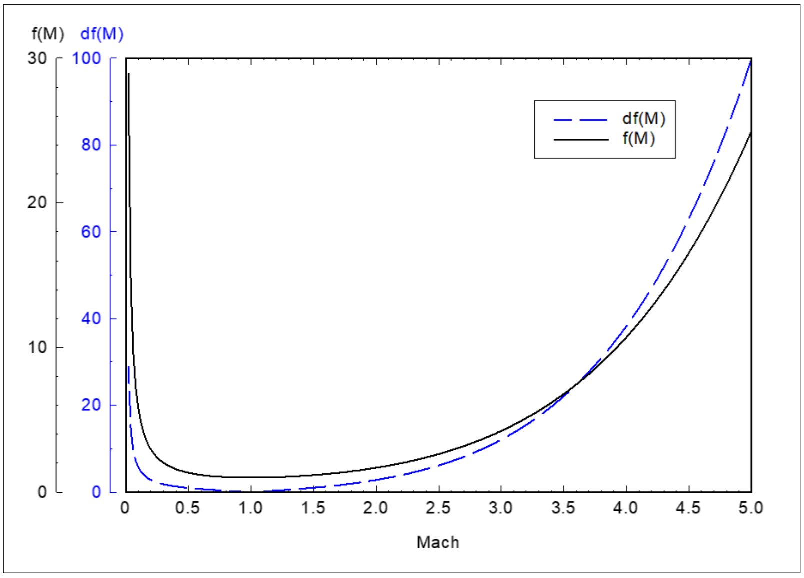


Fig. 4: Variation of Mach function and Mach gradient in the nozzle

2.5 The Area-deflection relation

The shocks become vanishingly weak, their limiting position being straight Mach lines. We can rewrite the differential equation in terms of Mach number M assuming the limit of smooth flow, *i.e.*, the velocities and flow inclinations must be continuous, but their derivatives may still be discontinuous with following expressions

$$\mathrm{du} = \mathrm{d}V \quad (18a)$$

$$\mathrm{dv} = \mathrm{V}\mathrm{d}\theta \quad (18b)$$

$$-\frac{du}{dv} = \tan\mu = \frac{1}{\sqrt{M^2-1}} \quad (18c)$$

where u and v are velocity components in the x- and y-Cartesian coordinates, respectively, and w is the resultant velocity. θ is inclination of a velocity vector. μ is Mach angle. Eliminating du and dv from this set of equations yields

$$\frac{1}{V}\frac{dV}{d\theta} = -\frac{1}{\sqrt{M^2-1}} \quad (19)$$

We consider simple-wave flow, Prandtl-Meyer flow or corner flow in which all the flow properties are uniform along each Mach line and Mach lines are straight. Mach lines are lines along which very weak disturbances propagate. For a given initial conditions, the magnitude of velocity at any point depends only on the flow direction at that point is

$$-\frac{d\theta}{dM} = \frac{\sqrt{M^2-1}}{M\left(1+\frac{\gamma-1}{2}M^2\right)} \quad (20)$$

The expansion at a corner as depicted in Fig. 2 occurs through a centered wave, defined by a "*fan*" of straight Mach lines. The flow up to the corner is uniform, at Mach M_1, and thus the leading Mach wave must be straight line, at the Mach angle μ_1. The terminating Mach line stands at the angle μ_2 to the downstream wall. Expansion is isentropic. From the geometry of the figure, noting that du is in limit the algebraic increment in velocity dV. The centered wave, more often called a Prandtl-Meyer expansion fan, is the counterpart for a convex corner as shown in Fig. (2a). The differential relation between θ and M in an isentropic compression or expansion [1] by turning, may be written as

$$-d\theta = \sqrt{M^2-1}\frac{dV}{V} \quad (21a)$$

or

$$\upsilon(M) = \int \sqrt{M^2-1}\frac{dV}{V} \quad (21b)$$

The function $\nu(M)$ is then

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma+1}{\gamma-1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1} \quad (21c)$$

$\nu(M)$ is called the Prandtl-Meyer function [1]. The constant of integration [1] has been arbitrarily chosen so that ν (M =1).

The Mach number at any point on the body surface is $\theta = \nu(M_1) - \nu(M_2)$

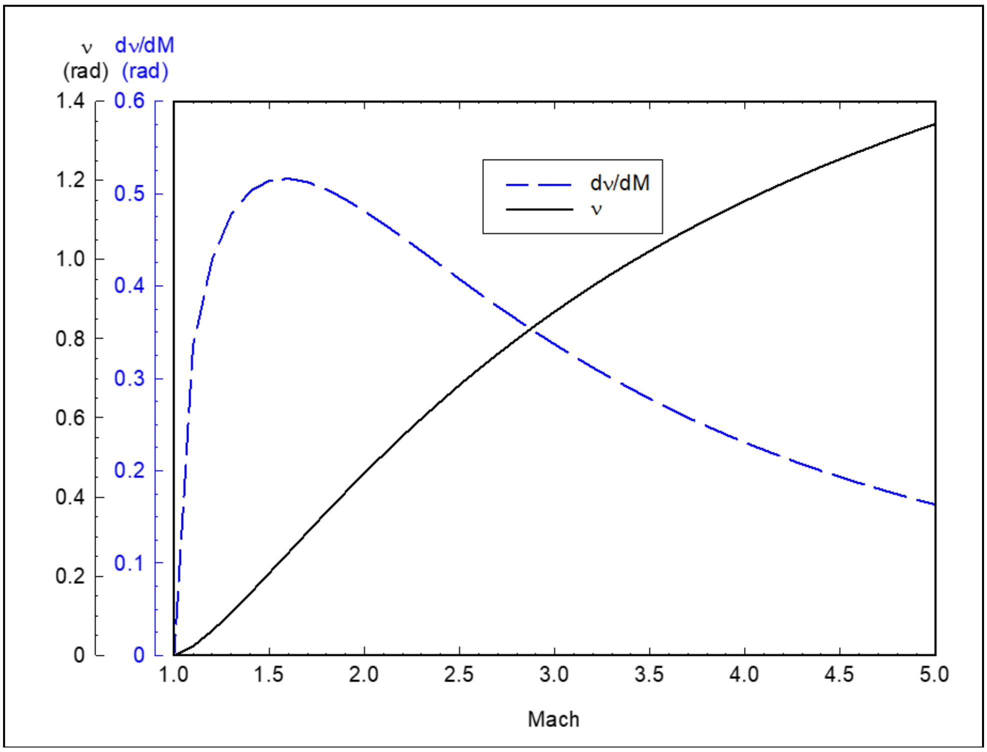


Fig. 4: Variation of the Prandtl-Meyer function and gradient versus Mach number

The variation of ν with M can also be seen in Figure 4. It is worth mentioning here that the Prandtl-Meyer function shows a monotonically increasing trend with the Mach number whereas [$d\nu/dM$] is increasing and reaches a maximum value at about $M = 1.6$ and after that a decreasing trend is observed with respect to Mach number. The method of characteristics can be applied to compute the Mach number for the calculated value of the Prandtl-Meyer function. Equations (11) and (14) are used to compute the Mach number employing Newton-Raphson method to get the value of M. $F'(M)$ is sensitivity to A/A^* of the conical nozzle. In the contour nozzle, $[d\nu/dM]$ is a measure of sensitivity of the estimated value of Mach number with respect to the Prandtl-Meyer function. $F'(M)$ and $[d\nu/dM]$ are very stiff in the vicinity of the throat region, which gives the convergence problem in Newton-Raphson scheme.

Flow with centered simple waves, and is often called corner-type flow, or Prandtl-Meyer flow. Substituting Equations (4) and (5) yield and expressing M in terms of θ, we can obtain fallowing differential equation

$$-\frac{dA}{A} = \sqrt{M^2 - 1}\, d\theta = \frac{d\theta}{\tan\theta} \qquad (23)$$

In the quasi-one-dimensional analyses the sign of $(M^2 - 1)$ was determining factor in the qualitative nature of flow.

$$-\frac{dA}{A} = \frac{d\theta}{\tan\theta} \qquad (24)$$

and the integration yields

$$\frac{A_1}{A_2} = \frac{\sin\theta_1}{\sin\theta_2} \qquad (25)$$

is obtained by summing infinitesimal contribution under a reversible process. The flow is isentropic. For the simple-wave flow is of Prandtl-Meyer type, we have

$$\frac{r}{r^*} = \left[\frac{2}{\gamma+1}\left\{1 + \frac{\gamma-1}{2}M^2\right\}\right]^{\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)} \qquad (26)$$

and the streamlines may be plotted with r and ν as polar coordinate [2].

2.6 The Area-pressure relation

Pressure variation dp across the Mach wave is found with the aid of the differential form of the equilibrium equation of Euler's equation. Because $dp/d\theta$ is a negative number, flow deflection as shown to Fig. 2 is accompanied by a decrease to pressure, that is, the fluid has been expanded. The Mach wave is an expansion wave. The relations derived here are valid for infinitesimal flow deflection. The disturbances caused by the deflection is infinitesimal and the condition of isentropic assumed is valid. Differential equation for pressure across the Mach wave [4] may be written as

$$\frac{dV}{V} = -\frac{1}{\gamma M^2}\frac{dp}{p} \qquad (27a)$$

Substituting Eq. (4) in Equation (27a), differential equation becomes

$$\frac{dp}{d\theta} = -\frac{\gamma M^2}{\sqrt{M^2-1}}p \qquad (27b)$$

Integration of Equation (27b) is

$$p^{\gamma} = tan\theta + c_2 \qquad (27c)$$

where c_2 is integration constant. After taking limit we can write as

$$\frac{p_1}{p_2} = -\left(\frac{\tan\theta_2}{\tan\theta_1}\right)^{\gamma} \qquad (27d)$$

The relation between area with pressure may be written in the form as obtained by Kuethe et al. [4]

$$\frac{A^*}{A} = \frac{\left[\left\{1-\left(\frac{p}{p_0}\right)^{\frac{(\gamma-1)}{\gamma}}\right\}\right]^{\frac{1}{2}}\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}}}{\frac{(\gamma-1)}{2}^{\frac{1}{2}}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)}} \qquad (28)$$

The above equation is obtained employing mass flow rate at throat in conjunction with isentropic relation [4] can be written as

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left\{1 + \frac{\gamma-1}{2}M^2\right\}\right]^{\frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)} = \frac{\left(\sqrt{\frac{\gamma-1}{2}}\right)\sqrt{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}}{\sqrt{\left[1-\left(\frac{p}{p_0}\right)^{\frac{(\gamma-1)}{\gamma}}\right]}\left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}}} \qquad (29)$$

A nomogram for convergent-divergent is drawn using the above analytical relations. Figure 5 shows nomogram of Laval nozzle at ratio of specific heats equal to 1.4 for air. The present nomogram can be reconstructed with the help of a computer program.

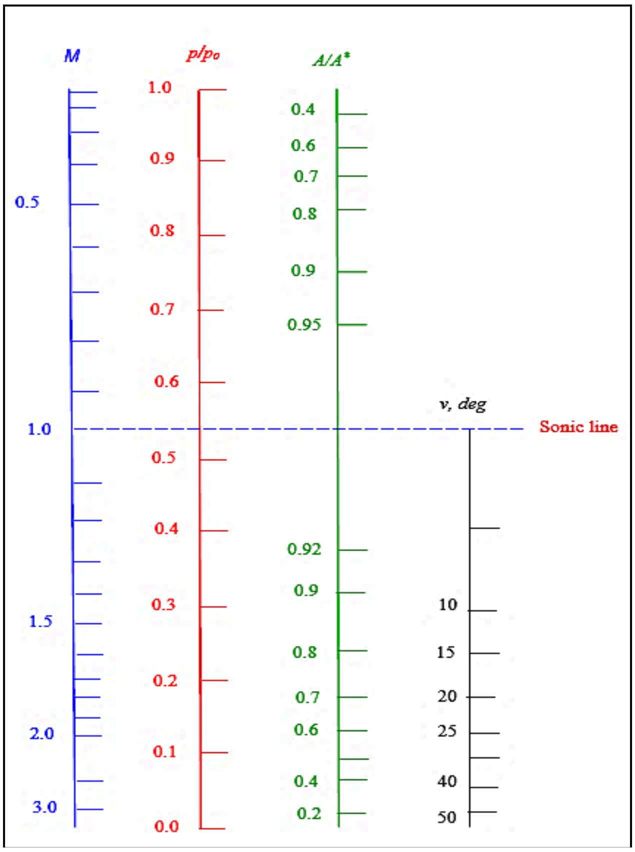


Fig. 5: Nomogram for convergent-divergent

RESULTS AND DISCUSSION

The relationship derived for the area-Mach, the area-Deflection and the area-Pressure for a fully expanded convergent-divergent Nozzle. However, it has no information on the length and nozzle contour of the rocket nozzle. Rao [10] developed a method of designing a contoured exhaust nozzle for optimum thrust of a fixed length rocket nozzle. Rao's [11] objective was to form nozzle contours that allowed for the maximization of thrust for a given rocket nozzle length.

CONCLUSIONS

Analytical expressions are obtained using steady compressible quasi-one-dimensional isentropic flow in fully expanded supersonic nozzles. Very weak disturbances are propagating along the Mach lines. The limit of smooth flow over a centred expansion wave is considered to obtain a relationship between area-Mach, area-deflection and pressure-deflection. Expansion fan is a continuous expansion region composed of an infinite number of Mach waves. An infinitesimal small flow deflection across a very weak wave and streamlines through an expansion wave are smooth lines. Nomogram for convergent-divergent nozzles is constructed using the present analytical solutions.

Nomenclature

a = speed of sound, m/s
A = area, m^2
D = diameter, m
M = Mach number, V/a
r = radius, m
u = axial velocity, m/s
V = resultant velocity, m/s
x = axial distance, m
θ = deflection angle, deg
ν = Prandtl-Meyer function
μ = Mach angle, deg
ρ = density, kg/m^3
γ = ratio of specific heats

Superscript
$*$ = sonic conditions
Subscript
o = stagnation conditions

REFERENCES

1. Liepmann, H. W. and Roshko, A., *Elements of Gas Dynamics*, First South Asian Edition, Dover Publications, Inc., New Delhi, India, 2007.
2. Shapiro, A. H., *The Dynamics and Thermodynamics of Fluid Flow*, Vol. 1, Ronald Press, New York, 1953.
3. Saad, M. A., *Compressible Fluid Flow*, Prentice-Hall, Inc., Eaglewood Cliffs, New Jersey, 1985.
4. Kuethe, A. M. and Chow, C-Y, *Foundations of Aerodynamics,* 5th Edition, John Wiley & Sons, New Delhi, India, 2010

5. Anderson, J. D., *Foundations of Aerodynamics*, International Student Edition, McGraw-Hill Book Company, Singapore, 1985.
6. Bertin, J. J., *Aerodynamics for Engineers*, Pearson Education, Inc. New Delhi, India, 2006.
7. Rathakrishnan, E., *Gas Dynamics*, 6th Edition, PHI Learning Limited, New Delhi, India, 2017
8. Mehta, R. C., Natarajan, G. and Bose, N. Fully expanded supersonic flow inside conical and contour nozzle, J. Spacecraft Rockets, 49(2), pp.422-425, 2012. doi.org/10.2514/1. A32195
9. Bose, T. K., *Computational Fluid Dynamics*, John Wiley & Sons, New Delhi, India, 1989.
10. Rao, G. V. R., Exhaust Nozzle contour for optimum thrust, Jet Propulsion, 28(6) 1958, pp. 377–382. doi:10.2514/8.7324
11. Rao, G. V. R., Approximation of optimum thrust nozzle contour, ARS Journal, 30, pp. 561, 1960. doi:10.2514/8.7324