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Research Article

Lattice Points on the Homogeneous Cone $59X^2 + Y^2 = Z^2$

K.Meena[1], S.Vidhyalakshmi [2], N.Sujitha[3], M.A.Gopalan[4*]
[1]Former VC, Bharathidasan University, Tiruchirappalli, India
[2,4]Professor, Department of Mathematics, Shrimati Indira Gandhi College, Tiruchirappalli, India
[3]Research Scholar in Mathematics, Shrimati Indira Gandhi College, Tiruchirappalli, India

***Corresponding author**
Dr. M. A.Gopalan
Email: mayilgopalan@gmail.com

Abstract: The ternary quadratic homogeneous equation representing homogeneous cone given by $59x^2 + y^2 = z^2$ by is analyzed for its non-zero distinct integer points on it. Three different patterns of integer points satisfying the cone under consideration are obtained. A few interesting relations between the solutions and special number patterns namely Polygonal number, Pyramidal number, Octahedral number, Pronic number, Stella Octangular number, Pentatope number and Nasty number are presented. Also knowing an integer solution satisfying the given cone, three triples of integers generated from the given solution are exhibited.
Keywords: Ternary homogeneous quadratic, integral solutions

INTRODUCTION

The ternary quadratic Diophantine equations offer an unlimited field for research due to their variety [1, 21]. For an extensive review of various problems, one may refer [2 - 20]. This communication concerns with yet another interesting ternary quadratic equation $59x^2 + y^2 = z^2$ representing a cone for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

Notations

P_n^m - Pyramidal number of rank n with size m.

$T_{m,n}$ - Polygonal number of rank n with size m.

$\Pr_n$ - Pronic number of rank n

OH_n - Octahedral number of rank n

SO_n - Stella octangular number of rank n

Pt_n - Pentatope number of rank n

METHOD OF ANALYSIS

The ternary quadratic equation under consideration is

$$59x^2 + y^2 = z^2 \quad (1)$$

To start with it is seen that the triples $(k, 29k, 30k), (2k+1, 2k^2+2k-29, 2k^2+2k+30)$ and $(2rs, r^2-59s^2, r^2+59s^2)$ satisfy (1).

However, we have other choices of solutions to (1) which are illustrated below
Consider (1) as

$$59x^2 + y^2 = z^2 * 1 \quad (2)$$

Assume

$$z = a^2 + 59b^2 \quad (3)$$

Write 1 as

$$1=\frac{\{(29+2n-2n^2)+i(2n-1)\sqrt{59}\}\{(29+2n-2n^2)-i(2n-1)\sqrt{59}\}}{(30-2n+2n^2)^2} \quad (4)$$

Substituting (3) and (4) in (2) and employing the method of factorization, define

$$y+i\sqrt{59}x=(a+i\sqrt{59}b)^2\left[\frac{[(29+2n-2n^2)+i(2n-1)\sqrt{59}]}{(30-2n+2n^2)^2}\right]$$

Equating the real and imaginary parts in the above equation, we get

$$x=\frac{1}{30-2n+2n^2}\left[2(29+2n-2n^2)ab-(a^2-59b^2)(2n-1)\right]$$

$$y=\frac{1}{30-2n+2n^2}\left[(29+2n-2n^2)(a^2-59b^2)-118ab(2n-1)\right]$$

$$z=a^2+59b^2$$

Replacing "a" by $(30-2n+2n^2)A$, "b" by $(30-2n+2n^2)B$ in the above equation corresponding integer solutions to (1) are given by

$$\left.\begin{aligned}x&=(30-2n+2n^2)[(A^2-59B^2)(2n-1)+\{2AB(29+2n-2n^2)\}]\\ y&=(30-2n+2n^2)[(A^2-59B^2)(29+2n-2n^2)-\{118AB(2n-1)\}]\\ z&=(30-2n+2n^2)^2(A^2+59B^2)\end{aligned}\right\}\text{(A)}$$

For simplicity and clear understanding, taking $n=1$ in (A), the corresponding integer solutions of (1) are given by

$$x=30A^2-1770B^2+1740AB$$

$$y=870A^2-1770B^2-3540AB$$

$$z=30^2(A^2+59B^2)$$

Properties

- $x(A,1)-t_{(62,A)}\equiv -1(\mathrm{mod}\,1769)$
- $x(A,1)-t_{(34,A)}-t_{(28,A)}\equiv -2(\mathrm{mod}\,1768)$
- $x(A,1)-2t_{(32,A)}\equiv -2(\mathrm{mod}\,1768)$
- $x(A,1)-t_{(28,A)}-t_{(36,A)}\equiv -2(\mathrm{mod}\,1768)$
- $y(A,1)-t_{(1742,A)}\equiv -581(\mathrm{mod}\,2671)$
- $y(A,1)-t_{(862,A)}-t_{(882,A)}\equiv -562(\mathrm{mod}\,2672)$
- $x(1,B)+t_{(3542,A)}\equiv 1(\mathrm{mod}\,29)$
- $y(1,B)+t_{(102662,B)}\equiv 870(\mathrm{mod}\,54869)$
- $y(1,B)+t_{(51322,B)}+t_{(51342,B)}\equiv 58(\mathrm{mod}\,54868)$
- $30x(A,1)-t_{(4,A)}\equiv -1(\mathrm{mod}\,58)$
- $30y(A,1)-t_{(60,A)}\equiv 1(\mathrm{mod}\,90)$
- $30x(1,b)+t_{(120,b)}=1$
- $30y(1,b)+t_{(3424,b)}\equiv 29(\mathrm{mod}\,1828)$
- $29x(A,1)-y(A,1)\equiv 0(\mathrm{mod}\,54000)$
- $29x(1,B)-y(1,B)\equiv 0(\mathrm{mod}\,54000)$
- $z(1,B)-t_{(4,B)}=59$
- $z(A,A+1)-t_{(122,A)}\equiv 1(\mathrm{mod}\,61)$
- $z(A,2A^2+1)-4t_{(4,A^2)}-t_{(128,A)}\equiv 1(\mathrm{mod}\,62)$

- $z(A,2A^2-1)-4t_{(4,A^2)}-t_{(112,A)}\equiv 1(\mod 54)$
- $z(A^2,(A+1))-59t_{(4,A^2)}-t_{(4,A)}\equiv 1(\mod 2)$
- $30y(A,2A^2-1)+6844t_{(4,A^2)}+6p_A^4-t_{(14456,A)}\equiv -1711(\mod 7462)$
- $30y(A,2A^2+1)+6844t_{(4,A^2)}+6p_A^4-t_{(13724,A)}\equiv -1711(\mod 6860)$
- $29x(A,A+1)-y(A,A+1)-t_{(108002,A)}\equiv 0(\mod 107999)$
- $230x(A,2A^2+1)-t_{(472,A)}+236t_{(4,A^2)}-174(OH_A)\equiv -59(\mod 234)$
- $30x(A,2A^2-1)-t_{(476,A)}+236t_{(4,A^2)}-58(SO_A)\equiv -59(\mod 236)$
- $30x(A,A+1)-P_{rA}+t_{(118,A)}\equiv -59(\mod 175)$
- $30x(A^2,A+1)-t_{(4,A^2)}+t_{(120,A)}-116P_A^5\equiv -59(\mod 176)$
- $30x(A(A+1),(A+2)(A+3))-1392Pt_A-P_{rA^2}+59t_{(4,A)}+1770P_A^4+t_{(2598,A)}\equiv -2124(\mod 4543)$
- $10z(A,A)$ is a Nasty number

It is worth to note that 1 in (2) may also be represented as

$$1=\frac{\{(59-4n^2)+i(4n)\sqrt{59}\}\{(59-4n^2)-i(4n)\sqrt{59}\}}{(59+4n^2)^2}$$

Following the analysis presented above, the corresponding integer solutions to (1) are found to be

$$\left.\begin{aligned} x&=(59+4n^2)[(A^2-59B^2)(4n)+\{2AB(59-2n^2)\}]\\ y&=(59+4n^2)[(A^2-59B^2)(59-4n^2)-\{472nAB\}]\\ z&=(59+4n^2)^2(A^2+59B^2)\end{aligned}\right\}\qquad (B)$$

For the sake of simplicity, taking $n=1$ in (B), the corresponding integer solution of (1) are given by

$$x=252A^2-14868B^2+6930AB$$
$$y=3465A^2-204435B^2-29736AB$$
$$z=63^2(A^2+59B^2)$$

Properties

- $x(A,1)-t_{(506,A)}\equiv -506(\mod 7181)$
- $55x(A,1)-4y(A,1)\equiv 0(\mod 1143450)$
- $55x(1,B)-4y(1,B)\equiv 0(\mod 1143450)$
- $55x(A^2,A+1)-4y(A^2,A+1)-t_{(1143452,A)}+3430350p_A^4\equiv 0(\mod 1143449)$
- $55x(A,A+1)-4y(A,A+1)+t_{(2286902,A)}\equiv 0(\mod 2286899)$
- $55x(A,2A^2-1)-4y(A,2A^2-1)+6860700p_A^4-t_{(6860702,A)}\equiv 0(\mod 5717249)$
- $55x(A,2A^2+1)-4y(A,2A^2+1)+6860700p_A^4-3430350t_{(4,A)}=0$
- $55x(A(A+1),(A+2)(A+3))-4y(A(A+1),(A+2)(A+3))+27442800pt_A\equiv 0$
- $63x(A,A+1)+t_{(246,A)}\equiv -236(\mod 483)$
- $63x(A^2,A+1)-55t_{(4,A^2)}+1416p_A^4+t_{(6020,A)}\equiv -3245(\mod 9262)$
- $3465x(A,1)-252y(A,1)\equiv 0(\mod 7938)$
- $3465x(A,A+1)-252y(A,A+1)-t_{(15878,A)}\equiv 0(\mod 15875)$

Generation of integer solutions

Let (x_0, y_0, z_0) be any given integer solution of (1)

Then, each of the following triples of integers satisfies (1)

Triple 1: (x_{n1}, y_{n1}, z_{n1})

$$x_{n1} = 119x_0 + 4y_0 - 16z_0$$
$$y_{n1} = 236x_0 + 9y_0 - 32z_0$$
$$z_{n1} = 944x_0 + 32y_0 - 127z_0$$

Triple 2: (x_{n2}, y_{n2}, z_{n2})

$$x_{n2} = \frac{1}{58}[\{59(-29)^n - (29)^n\}x_0 + \{(29)^n - (-29)^n\}z_0]$$
$$y_{n2} = 29^n y_0$$
$$z_{n2} = \frac{1}{58}[\{59(-29)^n - 59(29)^n\}x_0 + \{59(29)^n - (-29)^n\}z_0]$$

Triple 3: (x_{n3}, y_{n3}, z_{n3})

$$x_{n3} = \frac{1}{60}[\{59(-30)^n + (30)^n\}x_0 + \{(-30)^n - (30)^n\}y_0]$$
$$y_{n3} = \frac{1}{60}[\{59(-30)^n - 59(30)^n\}x_0 + \{59(30)^n + (-30)^n\}y_0]$$
$$z_{n3} = 22^n z_0$$

Triple 4: (x_{n4}, y_{n4}, z_{n4})

$$x_{n4} = 3^n x_0$$
$$y_{n4} = \frac{1}{6}[\{8(3)^n - 2(-3)^n\}y_0 + \{4(-3)^n - 4(3)^n\}z_0]$$
$$z_{n4} = \frac{1}{6}[\{4(3)^n - 4(-3)^n\}y_0 + \{8(-3)^n - 2(3)^n\}z_0]$$

CONCLUSION

In this paper, we have presented three different patterns of non-zero distinct integer solutions of the homogeneous cone given by $59x^2 + y^2 = z^2$. To conclude, one may search for other patterns of solution and their corresponding properties.

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