Scholars Journal of Engineering and Technology (SJET) **ISSN 2321-435X (Online)**
Sch. J. Eng. Tech., 2014; 2(1):14-18 **ISSN 2347-9523 (Print)**
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www.saspublisher.com

Research Article

Integer Points on the Hyperbola $x^2 - 6xy + y^2 + 4x = 0$

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Abstract: The binary quadratic equation $x^2 - 6xy + y^2 + 4x = 0$ representing hyperbola is considered. Different patterns of solutions are obtained. A few interesting recurrence relations satisfied by x and y are exhibited.
Keywords: binary quadratic, hyperbola, integer solutions

INTRODUCTION:

The binary quadratic equation offers an unlimited field for research because of their variety [1-5]. In this context one may also refer [6-19]. This communication concerns with yet another interesting binary quadratic equation $x^2 - 6xy + y^2 + 4x = 0$ for determining its infinitely many non-zero integral solutions. Also a few interesting relations are presented.

METHOD OF ANALYSIS

The hyperbola under consideration is

$$x^2 - 6xy + y^2 + 4x = 0 \qquad (1)$$

Different patterns of solutions for (1) are illustrated below:

Pattern: 1

Introducing the linear transformations $(X \neq T \neq 0)$,

$$x = X + T \text{ and } y = X - T \qquad (2)$$

In (1), it becomes

$$Y^2 = 2Z^2 - 1 \qquad (3)$$

Where, $Y = 4T + 1$ and $Z = 2X - 1$ (4)

The smallest positive integer solution of (3) is

$$Z_0 = 1 \text{ and } Y_0 = 1$$

To find the other solution of (3), consider the pellian equation

$$Y^2 = 2Z^2 + 1$$

whose general solution $(\overline{Y_n}, \overline{Z_n})$ is given by

$$\overline{Y_n} = \frac{1}{2}\left[\left(3 + 2\sqrt{2}\right)^{n+1} + \left(3 - 2\sqrt{2}\right)^{n+1}\right]$$

$$\overline{Z_n} = \frac{1}{2\sqrt{2}}\left[\left(3 + 2\sqrt{2}\right)^{n+1} - \left(3 - 2\sqrt{2}\right)^{n+1}\right]$$

Applying Brahmagupta Lemma between (Y_0, Z_0) and $(\overline{Y_n}, \overline{Z_n})$, the general solutions to (3) are given by,

$$Y_{n+1} = Y_0 Y_n + 2Z_0 Z_n$$

$$Z_{n+1} = Z_0 Y_n + Y_0 Z_n$$

In view of (4), we have

$$X_{n+1} = \frac{1}{2}(Y_n + Z_n + 1)$$

$$T_{n+1} = \frac{1}{4}(Y_n + 2Z_n - 1)$$

Employing (2), the values of x and y satisfying (1) are given by

$$x_{n+1} = \frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{n+2} + \left(3-2\sqrt{2}\right)^{n+2}\right] + \frac{1}{4} \quad , \quad n = 1,3,5,......$$

$$y_{n+1} = \frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] + \frac{3}{4} \quad , \quad n = 1,3,5,......$$

Properties

- $4x_{n+4} - 140x_{n+2} + 24x_{n+1} = -28$
- $6x_{n+2} - x_{n+1} - x_{n+3} = 1$
- $34x_{n+3} - x_{n+5} - x_{n+1} = 8$
- $6x_{n+4} - x_{n+3} - x_{n+5} = 1$
- $y_{n+5} - 34y_{n+3} + y_{n+1} = -24$
- $70y_{n+2} - 2y_{n+4} - 12y_{n+1} = 48$
- $y_{n+4} + y_{n+2} - 6y_{n+3} = -3$
- $y_{n+5} - 6y_{n+4} + y_{n+3} = 3$
- Each of the expressions represents a Nasty Number:
  - $48x_{2n} + 18$
  - $48y_{2n+2} - 24$
- Each of the expressions represents a cubical integer:
  - $8x_{3n+5} + 24x_{n+1} - 8$
  - $8y_{3n+3} + 24y_{n+1} - 24$
- Each of the expressions represents a bi-quadratic integer:
  - $8x_{4n+7} + 256x_{n+1}^2 - 128x_{n+1} + 12$
  - $8y_{4n+4} + 256y_{n+1}^2 - 384y_{n+1} - 136$

Note

Instead of (2), if we consider the linear transformations $(X \neq T \neq 0)$,

$$x = X - T \text{ and } y = X + T$$

Then, the corresponding integer solutions to (1) are obtained as,

$$x_{n+1} = \frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{n+2} + \left(3-2\sqrt{2}\right)^{n+2}\right] + \frac{3}{4} \quad , \quad n = 0,2,4,......$$

$$y_{n+1} = \frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] + \frac{1}{4} \quad , \quad n = 0,2,4,......$$

The recurrence relations satisfied by x and y are given by

$$x_{n+1} + 3 = 6x_{n+2} - x_{n+3}; \quad x_1 = 5, x_3 = 145$$

$$y_{n+1} + 1 = 6y_{n+2} - y_{n+3}; \quad y_1 = 1, y_3 = 25$$

Some numerical examples of x and y satisfying (1) is given in the following table:

n	x_{n+1}	y_{n+1}
0	5	1
2	145	25
4	4901	841
6	166465	28561

8	5654885	970225
10	192099601	32959081
12	6525731525	1119638521

From the above table relations observed are as follows:

- x_{n+1} and y_{n+1} are always odd
- y_{6n-5} and y_{6n-1} are perfect squares
- $6y_{6n-1}$ is a Nasty number
- $x_{6n-5} \equiv 0 \pmod 5$
- $y_{6n-3} \equiv 0 \pmod 5$
- $x_{6n-3} \equiv 0 \pmod 5$

Pattern: 2

Treating (1) as a quadratic in x and solving for x, we get

$$x = 3y - 2 \pm 2\sqrt{2y^2 - 3y + 1} \qquad (5)$$

Let $\alpha^2 = 2y^2 - 3y + 1$ (6)

Substituting $y = \frac{Y+3}{4}$ (7)

In (6), we have

$$Y^2 = 8\alpha^2 + 1$$

whose general solution is given by,

$$Y_n = \frac{1}{2}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] \qquad (8)$$

$$\alpha_n = \frac{1}{4\sqrt{2}}\left[\left(3+2\sqrt{2}\right)^{n+1} - \left(3-2\sqrt{2}\right)^{n+1}\right] \qquad (9)$$

From (7) and (8), we have

$$y_n = \frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] + \frac{3}{4} \qquad (10)$$

Substituting (9) and (10) in (5) and taking the positive sign, the corresponding integer solutions to (1) are given by

$$x_n = \frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{n+2} + \left(3-2\sqrt{2}\right)^{n+2}\right] + \frac{1}{4}, \quad n = 1,3,5,......$$

$$y_n = \frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] + \frac{3}{4}, \quad n = 1,3,5,......$$

Properties

- $48x_{2n+2}$ is a Nasty Number
- $8x_{3n+4} + 24x_n - 8$ is a Cubical integer
- $8x_{4n+6} + 256x_n^2 - 128x_n + 12$ is a Bi-quadratic integer
- Define $\beta = 4y_n - 3$ and $\gamma = x_n - 3y_n + 2$. Note that the pair (β, γ) satisfies the hyperbola $\beta^2 = 2\gamma^2 + 1$
- $2x_{2n} = (4y_n - 3)^2$

Also, taking the negative sign in (5), the other set of solutions to (1) is given by

$$x_n = \frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{n} + \left(3-2\sqrt{2}\right)^{n}\right] + \frac{1}{4}, \quad n = 1,3,5,......$$

$$y_n = \frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] + \frac{3}{4} \quad , \quad n = 1,3,5,......$$

In addition, the above two sets of solutions satisfy the following properties:

- $6y_{n+2} - y_{n+3} - y_{n+1} = 3$
- $140y_{n+1} - 4y_{n+3} - 24y_n = 84$
- $y_{n+2} + y_n - 6y_{n+1} = -3$
- $34y_{n+2} - y_{n+4} - y_n = 24$
- $x_{n+4} + x_{n+2} - 6x_{n+3} = -1$
- $70x_{n+1} - 2x_{n+3} - 12x_n = 14$
- $34x_{n+2} - x_{n+4} - x_n = 8$
- $x_n + x_{n+2} - 6x_{n+1} = -1$
- $48y_{2n+1} - 24$ is a Nasty Number
- $8y_{3n+2} + 24y_n - 24$ is a Cubical integer
- $8y_{4n+3} + 256y_n^2 - 384y_n - 136$ is a Bi-quadratic integer

Pattern: 3

Treating (1) as a quadratic in y and solving for y, we get

$$y = 3x \pm 2\sqrt{2x^2 - x} \qquad (11)$$

Let $\alpha^2 = 2x^2 - x$ (12)

Substituting $x = \frac{X+1}{4}$ (13)

In (12), we have

$$X^2 = 8\alpha^2 + 1$$

whose general solution is given by,

$$X_n = \frac{1}{2}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] \qquad (14)$$

$$\alpha_n = \frac{1}{4\sqrt{2}}\left[\left(3+2\sqrt{2}\right)^{n+1} - \left(3-2\sqrt{2}\right)^{n+1}\right] \qquad (15)$$

From (13) and (14), we have

$$x_n = \frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] + \frac{1}{4} \qquad (16)$$

Substituting (15) and (16) in (11) and taking the positive sign, the corresponding integer solutions to (1) are given by

$$x_n = \frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] + \frac{1}{4} \quad , \quad n = 0,2,4,......$$

$$y_n = \frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{n+2} + \left(3-2\sqrt{2}\right)^{n+2}\right] + \frac{3}{4} \quad , \quad n = 0,2,4,......$$

Properties

- $48y_{2n+2} - 24$ is a Nasty Number
- $8y_{3n+4} + 24y_n - 24$ is a Cubical integer
- $8y_{4n+6} + 256y_n^2 - 384y_n + 136$ is a Bi-quadratic integer
- $2y_{2n} - 1 = (4x_n - 1)^2$

Also, taking the negative sign in (11), the other set of solutions to (1) is given by

$$x_n = \frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right] + \frac{1}{4} \quad , \quad n = 0,2,4,......$$

$$y_n = \frac{1}{8}\left[\left(3+2\sqrt{2}\right)^{n} + \left(3-2\sqrt{2}\right)^{n}\right] + \frac{3}{4} \quad , \quad n = 0,2,4,......$$

In addition, the above two sets of solutions satisfy the following properties:

- $48x_{2n+1}$ is a Nasty Number
- $8x_{3n+2} + 24x_n - 8$ is a Cubical integer
- $8x_{4n+3} + 256x_n^2 - 128x_n + 12$ is a Bi-quadratic integer

CONCLUSION

As the binary quadratic equations are rich in variety, one may consider other choices of hyperbolas and search for their non-trivial distinct integral solutions along with the corresponding properties.

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