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Research Article

Some Results Of Fixed Point Theorem In Dislocated Quasi-Metric Spaces Of Integral Type

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Abstract: The purpose of this paper is to the study of fixed point theorems in dislocated quasi- metric spaces of integral type and obtain some new results in it. Also the paper contains generalized fixed point theorems of F. M. Zeyada et al., C.T. Aage & J.N. Salunke in dislocated quasi-metric space in integral type

Keywords: Fixed point theorem, Continuous Mapping, Complete metric space

INTRODUCTION

Let X be a nonempty set and let $d : X \times X \to [0,\infty)$ be a functiion satisfying the following conditions :

(i) $\int_o^{d(x,y)} \xi(t)dt = \int_o^{d(y,x)} \xi(t)dt = 0 \Rightarrow$ x = y

(ii) $\int_o^{d(x,y)} \xi(t)dt \le \int_o^{d(x,z)} \xi(t)dt + \int_o^{d(y,z)} \xi(t)dt =$, for all $x, y, z \in X$.

Then d is called a dislocated quasi-metric on X. If d satisfies $\int_o^{d(x,x)} \xi(t)dt = 0$, then it is called a quasi-metric on X. If d satisfies $d(x, y) = d(y, x)$, then it is called a dislocated metric.

Definition 1.1 *Let X be a nonempty set and $p : X \times X \to [0,\infty)$ be a function. We say p is a partial metric on X if it satisfies the following axioms:*

(i) $x = y$ *if and only if* $p(x, x) = p(x, y) = p(y, y)$, (ii) $p(x, x) \le p(x, y)$

(iii) $p(x, y) = p(y, x)$

(iv) $p(x, z) \le p(x, y) + p(y, z) - p(y, y)$
for all $x, y, z \in X$

Observe that any partial metric is a dislocated metric. Ultra metric d on X is a metric on X with condition $\int_o^{d(x,y)} \xi(t)dt \le \int_o^{d(x,z)} \xi(t)dt$, $\int_o^{d(z,y)} \xi(t)dt$. The study of partial metric spaces and generalized ultra metric spaces have application in theoretical computer science[2, 3]. The notion of the dislocated topologies is useful in the context of logic programming. Recently, Zeyada et al.[1] have established a fixed point theorem in a complete dislocated quasi-metric (dq-metric) space, as stated in the following lemma and theorem.

Lemma 1.1 *Let (X, d) be a dq-metric space. If $f : X \to X$ is a contraction function, then $\{(f^n(x_0))\}$ is a cauchy sequence for each $x_0 \in X$.*

Theorem 1.1 *Let (X, d) be a complete dq-metric space and let $f : X \to X$ be a continuous contraction function. Then f has a unique fixed point.*

PRELIMINARIES

Definition2.1 *A sequence $\{X_n\}$ in a dq-metric space (dislocated quasi-metric space) (X, d) is called Cauchy if for given $\in > 0, \exists n_0 \in N$ such that $\forall m, n \geq n_0$, implies*

$$\int_o^{d(x_n,x_m)} \xi(t)dt < \in \quad \text{or} \quad \int_o^{d(x_m,x_n)} \xi(t)dt < \in$$

i.e. $$\int_o^{mim\{d(x_n,x_m),d(x_m,x_n)\}} \xi(t)dt < \in$$

In the above definition if we replace $\int_o^{d(x_n,x_m)} \xi(t)dt < \in$ or $\int_o^{d(x_m,x_n)} \xi(t)dt < \in$

By $$\int_o^{\max\{d(x_n,x_m),d(x_m,x_n)\}} \xi(t)dt < \in,$$

the sequence $\{x_n\}$ is called "bi" Cauchy.Note that every bi Cauchy sequence is Cauchy.

Definition 2.2 *A sequence $\{X_n\}$ dislocated quasi-converges to x if*

$$\lim_{n\to\infty} \int_o^{d(x,x_n)} \xi(t)dt = \lim_{n\to\infty} \int_o^{d(x_n,x)} \xi(t)dt = 0$$

In this case x is called a dq-limit of $\{x_n\}$.

Proposition 2.1 . *Every convergent sequence in a dq-metric space is 'bi 'Cauchy.*

Proof. Let $\{x_n\}$ be a convergent sequence in a dq-metric space (X, d) and $x \in X$ be its dq-limit. That is,

$$\lim_{n\to\infty} \int_o^{d(x,x_n)} \xi(t)dt = \lim_{n\to\infty} \int_o^{d(x_n,x)} \xi(t)dt = 0$$

$\int_o^{d(x,x_n)} \xi(t)dt < \epsilon/2$. Now $n_0 = max\{ n_1; n_2 \} \in N$ is such that $m, n \geq n_0$ =) $\int_o^{d(x_n,x_m)} \xi(t)dt \leq \epsilon \cdot$

Then $\epsilon > 0; \exists n_1; n_2 \in N$ such that $n \geq n_1$ =) $\int_o^{d(x,x_m)} \xi(t)dt < \epsilon/2$ and $n \geq n_2$ =)

$\int_o^{d(x_n,x_m)} \xi(t)dt + \int_o^{d(x_m,x_n)} \xi(t)dt < \epsilon/2 + \epsilon/2 = \epsilon$ and $\int_o^{d(x_n,x_m)} \xi(t)dt \leq \int_o^{d(x,x_n)} \xi(t)dt \int_o^{d(x,x_m)} \xi(t)dt < \epsilon/2 + \epsilon/2 = \epsilon$

Hence $\{ x_n \}$ is bi Cauchy.

Converse of proposition 2.1 may not be true. Proof of the following lemma is obvious

Lemma 2.1 . *Every subsequence of dq-convergent sequence to a point x_0 is dq-convergent to x_0.*

Definition 2.3 *A dq-metric space* $(X; d)$ *is called complete if every Cauchy sequence in it is a dq-convergent.*

Definition 2.4 . *Let* (X, d_1) *and* (Y, d_2) *be dq-metric spaces and let* $f : X \to Y$ *be a function. Then* f *is continuous if for each sequence* $\{x_n\}$ *which is* d_1q*- convergent to* x_0 *in* X*, the sequence* $\{f(x_n)\}$ *is* d_2q*-convergent to* $f(x_0)$ *in* Y.

MAIN RESULTS

Theorem 3.1 *Let* (X, d) *be a complete dq-metric space and suppose there exist non negative constants* $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ *with* $\alpha_1+\alpha_2+\alpha_3+2(\alpha_4+\alpha_5) <1$. *Let* $f : X \to X$ *be a continuous mapping satisfying*

$$\int_0^{d(fx,fy)} \xi(t)dt \le \alpha_1 \int_0^{d(x,y)} \xi(t)dt + \alpha_2 \int_0^{d(x,fx)} \xi(t)dt + \alpha_3 \int_0^{d(y,fy)} \xi(t)dt + \alpha_4 \int_0^{d(x,fx)+d(y,fy)} \xi(t)dt + \alpha_5 \int_0^{d(x,fy)+d(y,fx)} \xi(t)dt$$

for all $x, y \in X$. *Then* f *has a unique fixed point.*

Proof: Let $\{x_n\}$ be a sequence in X, defined as follows. Let $x_0 \in X, f(x_0) = x_1, f(x_1) = x_2, \ldots, f(x_n) = x_{n+1}, \ldots$

$$\int_0^{d(Xn,Xn+1)} \xi(t)dt \le \int_0^{d(fXn-1,fXn)} \xi(t)dt$$

$$\le \alpha_1 \int_0^{d(Xn-1,Xn)} \xi(t)dt + \alpha_2 \int_0^{d(Xn-1,fXn-1)} \xi(t)dt + \alpha_3 \int_0^{d(Xn,fXn)} \xi(t)dt + \alpha_4 \int_0^{d(Xn-1,fXn-1)+d(Xn,fXn)} \xi(t)dt$$

$$\alpha_5 \int_0^{d(Xn-1,fXn)+d(Xn,fXn-1)} \xi(t)dt$$

$$= \alpha_1 \int_0^{d(Xn-1,Xn)} \xi(t)dt + \alpha_2 \int_0^{d(Xn-1,Xn)} \xi(t)dt + \alpha_3 \int_0^{d(Xn,Xn+1)} \xi(t)dt + \alpha_4 \int_0^{d(Xn-1,Xn)+d(Xn,Xn+1)} \xi(t)dt$$

$$\alpha_5 \int_0^{d(Xn-1,Xn+1)+d(Xn,Xn)} \xi(t)dt$$

$$\le (\alpha_1 + \alpha_2 + \alpha_4) \int_0^{d(Xn-1,Xn)} \xi(t)dt + (\alpha_3 + \alpha_4) \int_0^{d(Xn,Xn+1)} \xi(t)dt + \alpha_5 \int_0^{d(Xn-1,Xn)+d(Xn,Xn+1)} \xi(t)dt$$

$$= (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5) \int_0^{d(Xn-1,Xn)} \xi(t)dt + (\alpha_3 + \alpha_4 + \alpha_5) \int_0^{d(Xn,Xn+1)} \xi(t)dt$$

Therefore

$$\int_0^{d(Xn,Xn+1)} \xi(t)dt \le \lambda \int_0^{d(Xn-1,Xn)} \xi(t)dt \quad \text{where} \quad \lambda = \frac{\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5}{1 - \alpha_3 - \alpha_4 - \alpha_5}$$

Similly

$$\int_0^{d(Xn-1,Xn)} \xi(t)dt \le \lambda \int_0^{d(Xn-2,Xn-1)} \xi(t)dt$$

$$\int_0^{d(Xn,Xn+1)} \xi(t)dt \le \lambda^n \int_0^{d(Xo,X_1)} \xi(t)dt$$

Since $0 \le \lambda < 1$, so for $n \to \infty$, $\lambda^n \to \infty$ we have $d(x_n, x_{n+1}) \to 0$. Hence $\{x_n\}$ is a Cauchy sequence in the complete dislocated quasi-metric space X, so there is a point $t_0 \in X$, such that $x_n \to t_0$. Since f is continuous,

$$f(t_0) = \lim f(x_n) = \lim x_{n+1} = t_0.$$

Thus $f(t_0) = t_0$, so f has a fixed point.

Uniqueness: If $x \in X$ is a fixed point of f, then by (3.1)

$$\int_0^{d(x,x)} \xi(t)dt = \int_0^{d(fx,fx)} \xi(t)dt$$

$$\leq [\alpha_1 + \alpha_2 + \alpha_3 + 2(\alpha_4 + \alpha_5)] \int_0^{d(x,x)} \xi(t)dt$$

which is true only if $d(x, x) = 0$,since $0 \leq \alpha_1 + \alpha_2 + \alpha_3 + 2(\alpha_4 + \alpha_5) < 0$ and $d(x, x) \geq 0$. Thus $d(x, x) = 0$ for a fixed-point x of f.

Let x, y be fixed point of f. Then by (3.1)

$$\int_0^{d(x,y)} \xi(t)dt = \int_0^{d(fx,fy)} \xi(t)dt$$

$$\leq \alpha_1 \int_0^{d(x,y)} \xi(t)dt + \alpha_2 \int_0^{d(x,x)} \xi(t)dt + \alpha_3 \int_0^{d(y,y)} \xi(t)dt + \alpha_4 \int_0^{d(x,x)+d(y,y)} \xi(t)dt + \alpha_5 \int_0^{d(x,y)+d(y,x)} \xi(t)dt$$

$$\int_0^{d(x,y)} \xi(t)dt \leq (\alpha_1 + 2\alpha_5) \int_0^{d(x,y)} \xi(t)dt$$

and from this it follows that $d(x, y) = 0$, since $d(x, y) \geq 0$, $0 \leq (\alpha_1 + 2\alpha_5) < 1$. Similarly $d(y, x) = 0$. Hence $x = y$, i.e. uniqueness of the fixed point follows.

Note: If $\alpha_2 = 0 = \alpha_3$ in (3.1), then f becomes a contraction map and this shows that theorem 3.1 is a generalization of Theorem 1.1. Thus Theorem 3.1 is generalization of Banach contraction principle.

Theorem 3.2

Let (X, d) be a complete dq-metric space and let $f : X \to X$ be a continu- ous mapping satisfying

$$\int_0^{d(fx,fy)} \xi(t)dt \leq \alpha \int_0^{\max\{d(x,y),d(x,fx),d(y,fy)\}} \xi(t)dt + \beta \int_0^{\max\{d(x,fx)+d(y,fy),d(x,fy)+d(y,fx),d(x,y)\}} \xi(t)dt$$

for all $x, y \in X$. If $0 \leq \alpha, \beta < 1$ such that $\alpha + 2\beta < 1$ then f has a unique fixed point.

Proof: Let $\{x_n\}$ be a sequence in X, defined as follows. Let $x_0 \in X, f(x_0)=x_1, f(x_1)=x_2, \ldots, f(x_n)=x_{n+1}, \ldots$

$$\int_0^{d(Xn,Xn+1)} \xi(t)dt \le \int_0^{d(fXn-1,fXn)} \xi(t)dt$$

$$\le \alpha \int_0^{\max\{d(Xn-1,Xn),d(Xn-1,fXn-1),d(Xn,fXn)\}} \xi(t)dt + \beta \int_0^{\max\{d(fXn-1,Xn-1)+d(fXn,Xn),d(Xn-1,fXn)+d(Xn,fXn-1),d(Xn-1,Xn)\}} \xi(t)dt$$

$$\equiv \alpha \int_0^{\max\{d(Xn-1,Xn),d(Xn-1,Xn),d(Xn,Xn+1)\}} \xi(t)dt + \beta \int_0^{\max\{d(Xn,Xn-1)+d(Xn+1,Xn),d(Xn-1,Xn+1)+d(Xn,Xn),d(Xn-1,Xn)\}} \xi(t)dt$$

$$\equiv \alpha \int_0^{\max\{d(Xn-1,Xn),d(Xn,Xn+1)\}} \xi(t)dt + \beta \int_0^{\max\{d(Xn-1,Xn+1),d(Xn-1,Xn)\}} \xi(t)dt$$

Case-1

When $\int_0^{d(Xn-1,Xn)} \xi(t)dt = \int_0^{\max\{d(Xn-1,Xn+1),d(Xn-1,Xn)\}} \xi(t)dt$

$$\int_0^{d(Xn,Xn+1)} \xi(t)dt \le (\alpha+\beta) \int_0^{d(Xn-1,Xn)} \xi(t)dt$$

$$= \lambda \int_0^{d(Xn-1,Xn)} \xi(t)dt \quad \text{Where} \quad \lambda = \alpha + \beta$$

Similly $\int_0^{d(Xn-1,Xn)} \xi(t)dt \le \lambda \int_0^{d(Xn-2,Xn-1)} \xi(t)dt$

$$\int_0^{d(Xn,Xn+1)} \xi(t)dt \le \lambda^2 \int_0^{d(Xn-2,Xn-1)} \xi(t)dt$$

Thus $\int_0^{d(Xn,Xn+1)} \xi(t)dt \le \lambda^n \int_0^{d(X_1,X_0)} \xi(t)dt$

Since $0 \le \gamma < 1$, as $n \to \infty$, $\gamma^n \to \infty$. Hence $\{x_n\}$ isa dq-cauchy sequence in X. Thus $\{x_n\}$ dislocated quasi-converges to some t_0. Since f is continuous, we have $f(t_0) = \lim f(x_n) = \lim x_{n+1} = t_0$

Thus $f(t_0) = t_0$ that is f has a fixed point t_0.

Case-2

When $\int_0^{\max\{d(Xn-1,Xn+1),d(Xn-1,Xn)\}} \xi(t)dt = \int_0^{d(Xn-1,Xn+1)} \xi(t)dt$

$$\le \int_0^{d(Xn-1,Xn)+d(Xn,Xn+1)} \xi(t)dt$$

$$\int_0^{d(Xn,Xn+1)} \xi(t)dt \le \alpha \int_0^{d(Xn-1,Xn)} \xi(t)dt + \beta \int_0^{d(Xn-1,Xn)+d(Xn,Xn+1)} \xi(t)dt$$

$$(1-\beta) \int_0^{d(Xn,Xn+1)} \xi(t)dt \le (\alpha+\beta) \int_0^{d(Xn-1,Xn)} \xi(t)dt$$

$$\int_0^{d(Xn,Xn+1)} \xi(t)dt \le \left(\frac{\alpha+\beta}{1-\beta}\right) \int_0^{d(Xn-1,Xn)} \xi(t)dt$$

$$\int_0^{d(Xn,Xn+1)} \xi(t)dt \le \delta \int_0^{d(Xn-1,Xn)} \xi(t)dt \qquad \text{Where} \quad \delta = \tfrac{\alpha+\beta}{1-\beta} < 1$$

Uniqueness: Let *x* be a fixed point of *f*, then by (3.2)

$$\int_0^{d(x,x)} \xi(t)dt = \int_0^{d(fx,,fx)} \xi(t)dt \le \lambda' \int_0^{\max d(x,x)} \xi(t)dt \qquad \text{Where} \quad \lambda' = \alpha + 2\beta$$

i.e. $\int_0^{d(x,x)} \xi(t)dt \le \lambda' \int_0^{d(x,x)} \xi(t)dt$

which gives $d(x;x) = 0$, since $0 \le \gamma < 1$ and $d(x;x) \ge 0$. Thus

$d(x;x) = 0$ if x is a fixed point of f.

Let $x; y \in X$ be fixed points of f. That is ,$fx= x; fy= y$. Then by (3.2),

$$\int_0^{d(x,y)} \xi(t)dt = \int_0^{d(fx,,fy)} \xi(t)dt \le \alpha \int_0^{\max\{d(x,y),d(x,x),d(y,y)} \xi(t)dt + \beta \int_0^{\max\{d(x,x)+d(y,y),d(x,y)+d(y,x),d(x,y)\}} \xi(t)dt$$

$$= (\alpha + 2\beta) \int_0^{d(x,y)} \xi(t)dt$$

which is true only if $d(x;y) = 0$ since $d(x;x) = 0 = d(y;y)$; $0 \le \gamma < 1$.

Similarly $d(y;x) = 0$ and hence $x = y$.

Thus a ¯fixed point of *f* is unique

Note: If *d* is a partial metric on *X* , then (*X; d*) becomes a dq-metric space. Hence we consider (*X; d*) in Theorem 3.1 and 3.2 as a partial metric space, then the conclusion follows.

..

Theorem 3.3 *Let* (*X; d*) *be a complete partial metric space and let* $f: X \to X$ *be a continuous mapping satisfying*

$$\int_0^{d(fx,,fy)} \xi(t)dt \le \alpha \int_0^{\max\{d(x,y),d(x,fx),d(y,fy),d(y,fx)\}} \xi(t)dt + \beta \int_0^{\max\{d(fx,x)+d(fy,y),d(x,fy)+d(y,fx),d(x,y)\}} \xi(t)dt$$

for all $x; y \in X$. *If* $0 \le \alpha, \beta < 1$ *such that* $\alpha + 2\beta < 1$ *then f has a unique fixed point.*

It can be proved easily.

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