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Research Article

On The Ternary Quadratic Diophantine Equation $6(x^2+y^2)-8xy=21z^2$

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Abstract: The ternary quadratic equation $6(x^2+y^2)-8xy=21z^2$ representing a homogeneous cone is analyzed for its non-zero distinct integral points. A few interesting properties among the solutions and special numbers namely, Polygonal numbers, Centered polygonal numbers, Pyramidal numbers, Gnomonic numbers are presented.
Keywords: ternary quadratic, homogeneous cone, integer points

INTRODUCTION

The ternary homogeneous quadratic Diophantine equation offers an unlimited field for research because of their variety [1-2]. For an extensive review of various problems one may refer [3 – 11]. In this context, one may also see [12 – 8] for integer point satisfying special three dimensional graphical representations. This communication concerns with yet another interesting ternary quadratic equation $6(x^2+y^2)-8xy=21z^2$ representing homogeneous cone for determining its infinitely many non-zero integer solutions.

A few interesting properties among the solutions and special numbers are presented.

Notations Used:

$t_{m,n}$ - Polygonal number of rank n with size m

P_n^m - Pyramidal number of rank n with size m

$CP_{m,n}$ - Centered Polygonal number of rank n with size m.

GNO_n - Gnomonic number of rank n.

METHOD OF ANALYSIS

The Ternary Quadratic Diophantine Equation representing homogeneous cone is

$$6(x^2+y^2)-8xy=21z^2 \tag{1}$$

We illustrate below the different patterns of integer solutions to (1)

Pattern I:

Introducing the linear transformations

$$x=u+v, y=u-v \tag{2}$$

in (1), it is written as

$$(2u)^2+20v^2=21z^2 \tag{3}$$

Assume that

$$z=a^2+5b^2, a,b\neq 0 \tag{4}$$

and write 21 as

$$21 = (4 + i\sqrt{5})(4 - i\sqrt{5}) \quad (5)$$

Using (4) & (5) in (3) and applying the method of factorization, define

$$(2u + i2\sqrt{5}v) = (4 + i\sqrt{5})(a + i\sqrt{5}b)^2$$

Equating the real and imaginary parts we have

$$u(a,b) = 2a^2 - 10b^2 - 5ab$$

$$v(a,b) = \frac{1}{2}[a^2 + 8ab - 5b^2]$$

Then using (2) the values of x and y are given by

$$x(a,b) = \frac{1}{2}[5a^2 - 25b^2 - 2ab]$$

$$y(a,b) = \frac{1}{2}[3a^2 - 15b^2 - 18ab]$$

As our aim is to find integer solutions, choose a and b so that x and y are integers.

Case (I):

Choosing $a = 2A, b = 2B$ corresponding integral values x, y and z satisfying the homogeneous cone (1) are given by,

$$x = x(A,B) = 10A^2 - 50B^2 - 4AB$$

$$y = Y(A,B) = 6A^2 - 30B^2 - 36AB$$

$$z = z(A,B) = 4A^2 + 20B^2$$

Properties:

1. $x(A,1) + y(A,1) + z(A,1) - 20t_{4,A} + 60 \equiv 0 \pmod 4$
2. $x(A(A+1),A) + y(A(A+1),A) + 80t_{4,A} + 80P_A^{\,5}$ is a perfect square.
3. $x(A, A(A+1)) + Z(A, A(A+1)) + 120t_{3,A}^{\,2} + 8P_A^{\,5} = 14t_{4,A}$
4. $x(A,1) + y(A,1) + 20GNO_A + 100 = 16t_{4,A}$
5. $6[x(A(A+1),A) + y(A(A+1),A)] + 480(t_{4,A} + P_A^{\,5})$ Is a Nasty number.

Case (II):

Choosing $a = 2A+1, b = 2B+1$ corresponding the integral values x, y and z satisfying the homogeneous cone (1) are given by,

$$x = x(A,B) = 10A^2 - 50B^2 + 8A - 52B - 4AB - 11$$

$$y = y(A,B) = 6A^2 - 30B^2 - 36AB - 12A - 48B - 15$$

$$z = z(A,B) = 4A^2 + 20B^2 + 4A + 20B + 6$$

Properties:

1. $x(A,1) - y(A,1) - 8t_{3,A} + 20 \equiv 0 \pmod 8$

2. $x(A,1)-y(A,1)+z(A,1)-CP_{16,A}-159\equiv 0(\mathrm{mod}\,12)$
3. $y(A,A(A+1))+120t_{3,A}{}^{2}+72P_{A}{}^{5}+48t_{3,A}-6t_{4,A}+15\equiv 0(\mathrm{mod}\,5)$
4. $y(A,A)+120t_{3,A}+15=0$
5. $y(A,A)+x(A,A)+208t_{3,A}+26=0$

Note:

Instead of (5) we can also write 21as

$$21=(-4+i\sqrt{5})(-4-i\sqrt{5})$$

Preceding as above in Pattern I, we can get different choices of integer solutions to (1).

Pattern II:

Introducing the linear transformations

$$z=X+20T,v=X+21T \tag{6}$$

in (2), it is written as

$$X^{2}=420T^{2}+(2u)^{2} \tag{7}$$

which is satisfied by

$$T=2pq,2u=420p^{2}-q^{2},X=420p^{2}+q^{2}$$

Now putting $p=P$ and $q=2Q$ in the above we get,

$$T=4PQ$$
$$u=210P^{2}-2Q^{2} \tag{8}$$
$$X=420P^{2}+4Q^{2}$$

Using (6), (7) & (8) in (2) we get solutions of (1) as

$$x=x(P,Q)=630P^{2}+2Q^{2}+84PQ$$
$$y=y(P,Q)=-(210P^{2}+6Q^{2}+84PQ)$$
$$z=z(P,Q)=420P^{2}+4Q^{2}+80PQ$$

Properties:

1. $x(P,1)+y(P,1)-20t_{4,P}+4$ is a Perfect Square.
2. $6[x(P,1)+y(p,1)]-120t_{4,P}+24$ is a Nasty number.
3. $y(1,Q)+z(1,Q)-2t_{4,Q}+8t_{3,Q}=210$
4. $x(P,1)+y(P,1)+z(P,1)+10=760t_{4,P}+10CP_{16,A}$
5. $y(P,1)+z(P,1)-210t_{4,P}=2GNO_{P}$

Pattern III:

Write (3) as

$$(2u)^{2}-z^{2}=20(z^{2}-v^{2})$$

Factorizing the above, it is written as,

$\dfrac{(2u-z)}{4(z+v)}=\dfrac{5(z-v)}{(2u+z)}=\dfrac{A}{B},(B\neq 0)$ which is equivalent to the following

two equations, $$-2Bu+4Av+(4A+B)Z=0 \tag{9}$$

$$2Au+5Bv-(5B-A)Z=0 \tag{10}$$

By applying the method of cross multiplication, we get the integral solutions of (1) to be

$$x = x(A,B) = 12A^2 - 36AB - 15B^2$$
$$y = y(A,B) = 5B^2 - 4A^2 - 44AB$$
$$z = z(A,B) = -(8A^2 + 10B^2)$$

Properties:

1. $x(A,1) - y(A,1) + z(A,1) + 30 = 16t_{3,A}$
2. $x(A, A, (A+1)) + y(A, A(A+1)) + 160P_A^5 + 40t_{3,A}^2 - 8t_{4,A} = 0$
3. $x(A,1) + y(A,1) - z(A,1) - 16t_{4,A} \equiv 0 \pmod 4$
4. $x(1,B) + y(1,B) + 20t_{3,B} - 8 \equiv 0 \pmod 7$
5. $10[x(B, B+1) + y(B, B+1)] - 8t_{4,B} + 160t_{3,B}]$ Is a Perfect Square.

Pattern IV:

Also (3) is expressed in the form ratio as

$$\frac{(2u+z)}{(z+v)} = \frac{20(z-v)}{(2u-z)} = \frac{A}{B}, B \neq 0 \tag{11}$$

Following the procedure as in Pattern III, the corresponding solutions of (1) are as follows:

$$x = x(A,B) = 20B^2 + 44AB - A^2$$
$$y = y(A,B) = 3A^2 - 60B^2 + 36AB$$
$$z = z(A,B) = 2A^2 + 40B^2$$

Properties:

1. $x(A,1) + y(A,1) - 4t_{3,A} + 40 \equiv 0 \pmod 6$
2. $x(A,1) + y(A,1) + z(A,1) - 8t_{3,A} \equiv 0 \pmod 4$
3. $x(1,B) - 40t_{3,B} + 1 \equiv 0 \pmod 4$
4. $x(A,1) + y(A,1) + z(A,1) + 400$ is a Perfect Square.
5. $7[x(A+1, A) + y(A+1, A)] + 280$ is a Nasty number.

CONCLUSION

It is worth to note here that, in addition to (10) , equation (3) may also be written in the form of ratio as follows

$$\frac{(2u-z)}{5(z+v)} = \frac{4(z-v)}{(2u+z)}, \quad \frac{(2u+z)}{10(z+v)} = \frac{2(z-v)}{(2u-z)},$$

$$\frac{(2u+z)}{2(z+v)} = \frac{10(z-v)}{(2u-z)}, \quad \frac{(2u-z)}{20(z+v)} = \frac{(z-v)}{(2u+z)}$$

Following the procedure presented in Pattern III, One may arrive at different patterns of solutions to (1).

To conclude, one may search for other pattern of solutions and their corresponding properties.

Remarkable Observations:

1. If D represents the diagonal of any rectangle with dimensions x and y ; A its area then $3D^2 - 4A \equiv 0 \pmod{21}$
2. If (x_0, y_0, z_0) is any given solutions of (1), the triple $(21x_0 - 20y_0 - 42z_0, -20x_0, +21y_0 + 42z_0, 20x_0 - 20y_0 - 41z_0)$ also satisfies (1)
3. Employing the solutions (x, y, z) of (1) following relations among the special polygonal and pyramidal numbers is obtained.

(i) $$6\left[\left(\frac{P_x{}^5}{t_{3,x}}\right)^2+\left(\frac{3P^3{}_{y-2}}{t_{3,y-2}}\right)^2\right]-8\left[\left(\frac{P_x{}^5}{t_{3,x}}\right)\left(\frac{3P_{y-2}{}^3}{t_{3,y-2}}\right)\right]=21\left(\frac{2P_{z-1}}{t_{4,z-1}}\right)^2$$

(ii) $$6\left[\left(\frac{3P_{x-2}{}^3}{t_{3,x-2}}\right)^2+\left(\frac{2P_{y-1}}{t_{4,y-1}}\right)^2\right]-8\left[\left(\frac{3P_{x-2}{}^3}{t_{3,x-2}}\right)\left(\frac{2P_{y-1}}{t_{4,y-1}}\right)\right]=0(\mathrm{mod}\,21)$$

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