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Research Article

Integer Points on the Hyperbola $x^2 - 5xy + y^2 + 3x = 0$

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Abstract: The binary quadratic equation $x^2 - 5xy + y^2 + 3x = 0$ representing hyperbola is considered. Different patterns of solutions are obtained. A few interesting relations satisfied by x and y are exhibited.
Keywords: binary quadratic, hyperbola, integer solutions.

INTRODUCTION:

The binary quadratic equation offers an unlimited field for research because of their variety [1-5]. In this context one may also refer [6-20]. This communication concerns with yet another interesting binary quadratic equation $x^2 - 5xy + y^2 + 3x = 0$ representing hyperbola for determining its infinitely many non-zero integral solutions. Also a few interesting relations among the solutions are presented.

METHOD OF ANALYSIS:

The hyperbola under consideration is

$$x^2 - 5xy + y^2 + 3x = 0 \quad (1)$$

To start with, it is seen that (1) is satisfied by (1,1),(1,4),(16,4) and (-75,-15). However, we have other patterns of solutions for (1) which are illustrated below:

PATTERN: 1

Treating (1) as a quadratic in y and solving for y, we get

$$y = \frac{1}{2}[\,5x \pm \sqrt{21x^2 - 12x}\,] \quad (2)$$

Let $\alpha^2 = 21x^2 - 12x$ (3)

Substituting $x = \frac{X+2}{7}$ (4)

in (3), we have

$$3X^2 - 7\alpha^2 = 12 \quad (5)$$

Introducing the linear transformations

$$X = p + 7q \text{ and } \alpha = p + 3q \quad (6)$$

in (5), we have

$$p^2 = 21q^2 - 3 \quad (7)$$

which is satisfied by $q_0 = 2, p_0 = 9$

To find the other solutions of (7), consider the pellian equation

$$p^2 = 21q^2 + 1$$

whose general solution $(\overline{p_n}, \overline{q_n})$ is given by

$$\left.\begin{aligned}\overline{p_n} &= \frac{1}{2}\left[\left(55+12\sqrt{21}\right)^{n+1}+\left(55-12\sqrt{21}\right)^{n+1}\right]\\ \overline{q_n} &= \frac{1}{2\sqrt{21}}\left[\left(55+12\sqrt{21}\right)^{n+1}-\left(55-12\sqrt{21}\right)^{n+1}\right]\end{aligned}\right\} \quad (8)$$

Applying Brahmagupta Lemma between (p_0, q_0) and $(\overline{p_n}, \overline{q_n})$, the general solutions to (7) are given by,

$$\left.\begin{aligned} q_{n+1} &= 2\overline{p}_n + 9\overline{q}_n \\ p_{n+1} &= 9\overline{p}_n + 42\overline{q}_n \end{aligned}\right\} \quad (9)$$

In view of (6), we have

$$X_{n+1} = 23\overline{p}_n + 105\overline{q}_n \quad (10)$$

$$\alpha_{n+1} = 15\overline{p}_n + 69\overline{q}_n \quad (11)$$

Substituting (10) in (4), we get

$$x_{n+1} = \frac{1}{7}\{23\overline{p}_n + 105\overline{q}_n + 2\} \quad (12)$$

Substituting (11) and (12) in (2) and taking the positive sign, it is seen that

$$y_{n+1} = \frac{1}{14}\{220\overline{p}_n + 1008\overline{q}_n + 10\} \quad (13)$$

Substituting (8) in (12) and (13), the corresponding integer solutions to (1) are given by

$$14x_{2n+1} - 4 = 23\, f_{2n+1} + \frac{105}{\sqrt{21}} g_{2n+1}\ , \quad n = 0,1,2,3,\dots$$

$$7y_{2n+1} - 5 = 55\, f_{2n+1} + \frac{252}{\sqrt{21}} g_{2n+1}\ , \quad n = 0,1,2,3,\dots$$

where

$$f_{2n+1} = \left[\left(55+12\sqrt{21}\right)^{2n+1} + \left(55-12\sqrt{21}\right)^{2n+1}\right]$$

$$g_{2n+1} = \left[\left(55+12\sqrt{21}\right)^{2n+1} - \left(55-12\sqrt{21}\right)^{2n+1}\right]$$

PROPERTIES:

- $x_{2n+5} - 12098x_{2n+3} + x_{2n+1} = -3456$
- $12098x_{2n+3} - x_{2n+5} - x_{2n+1}$ is a Nasty number
- $y_{2n+5} - 12098y_{2n+3} + y_{2n+1} = -8640$
- $-10y_{2n+5} + 120980y_{2n+3} - 10y_{2n+1}$ is a Nasty number
- $7(12099y_{2n+1} - y_{2n+3}) + 14(12121x_{2n+1} - x_{2n+3}) - 108970 \equiv 0 \pmod{1166}$
- $7x_{2n+3} + 3857x_{2n+1} - 18480y_{2n+1} + 12096 = 0$
- $7y_{2n+3} + 18480x_{2n+1} - 88543y_{2n+1} + 57960 = 0$

Now,

Substituting (11) and (12) in (2) and taking the negative sign, the corresponding integer solutions to (1) are given by

$$14x_{2n+1} - 4 = 23\, f_{2n+1} + \frac{105}{\sqrt{21}} g_{2n+1}\ , \quad n = 0,1,2,3,\dots$$

$$14y_{2n+1} - 10 = 5\, f_{2n+1} + \frac{21}{\sqrt{21}} g_{2n+1}\ , \quad n = 0,1,2,3,\dots$$

where

$$f_{2n+1} = \left[\left(55+12\sqrt{21}\right)^{2n+1} + \left(55-12\sqrt{21}\right)^{2n+1}\right]$$

$$g_{2n+1} = \left[\left(55+12\sqrt{21}\right)^{2n+1} - \left(55-12\sqrt{21}\right)^{2n+1}\right]$$

PROPERTIES:

- $84847x_{2n+1} - 7x_{2n+3} - 24240 \equiv 0 \pmod{528}$
- $x_{2n+3} + 2640y_{2n+1} - 12649x_{2n+1} + 1728 = 0$
- $y_{2n+3} + 551y_{2n+1} - 2640x_{2n+1} + 360 = 0$
- $26400x_{2n+1} - 5510y_{2n+1} - 10y_{2n+3}$ is a perfect square
- $25298x_{2n+1} - 5280y_{2n+1} - 2x_{2n+3}$ is a Nasty number

NOTE:

Instead of (6), if we consider the linear transformations

$X = p - 7q$ and $\alpha = p - 3q$

then, the corresponding two sets of integer solutions to (1) are obtained as

Set I:

$$14x_{2n+1} - 4 = -5f_{2n+1} - \frac{21}{\sqrt{21}}g_{2n+1}, \quad n = 0,1,2,3,\ldots$$

$$14y_{2n+1} - 10 = -23f_{2n+1} - \frac{105}{\sqrt{21}}g_{2n+1}, \quad n = 0,1,2,3,\ldots$$

Set II:

$$7x_{2n+1} - 5 = -f_{2n+1}, \quad n = 0,1,2,3,\ldots$$

$$14y_{2n+1} - 10 = -23f_{2n+1} - \frac{105}{\sqrt{21}}g_{2n+1}, \quad n = 0,1,2,3,\ldots$$

where

$$f_{2n+1} = \left[\left(55+12\sqrt{21}\right)^{2n+1} + \left(55-12\sqrt{21}\right)^{2n+1}\right]$$

$$g_{2n+1} = \left[\left(55+12\sqrt{21}\right)^{2n+1} - \left(55-12\sqrt{21}\right)^{2n+1}\right]$$

PATTERN: 2

Treating (1) as a quadratic in x and solving for x, we get

$$x = \frac{1}{2}[\, 5y - 3 \pm \sqrt{9 + 21y^2 - 30y}\,] \qquad (14)$$

Let $\alpha^2 = 9 + 21y^2 - 30y$ (15)

Substituting $y = \frac{Y+5}{7}$ (16)

in (15), we have

$$3Y^2 - 7\alpha^2 = 12 \qquad (17)$$

Introducing the linear transformations

$$Y = p + 7q \text{ and } \alpha = p + 3q \qquad (18)$$

Following the analysis similar to pattern 1, the two sets (set III,set IV) of integer solutions to (1) are represented by

Set III:

$$7x_{2n+1} - 11 = 55f_{2n+1} + \frac{252}{\sqrt{21}}g_{2n+1}, \quad n = 0,1,2,3,\ldots$$

$$14y_{2n+1} - 10 = 23f_{2n+1} + \frac{105}{\sqrt{21}}g_{2n+1}, \quad n = 0,1,2,3,\ldots$$

where

$$f_{2n+1}=\left[\left(55+12\sqrt{21}\right)^{2n+1}+\left(55-12\sqrt{21}\right)^{2n+1}\right]$$

$$g_{2n+1}=\left[\left(55+12\sqrt{21}\right)^{2n+1}-\left(55-12\sqrt{21}\right)^{2n+1}\right]$$

PROPERTIES:

- $x_{2n+5}-12098x_{2n+3}+x_{2n+1}=-19008$
- $266156x_{2n+3}-22x_{2n+5}-22x_{2n+1}$ is a Nasty number
- $14(y_{2n+5}-146639857y_{2n+1})-7(x_{2n+3}-12099x_{2n+1})+1466265482\equiv 0(\bmod 12775378)$
- $7(x_{2n+5}-146373701x_{2n+1})+1610110700\equiv 0(\bmod 1330780)$
- $7y_{2n+3}+3857y_{2n+1}-18480x_{2n+1}+26280=0$
- $7x_{2n+3}+18480y_{2n+1}-88543x_{2n+1}+113829=0$

Set IV:

$$14x_{2n+1}-22=5f_{2n+1}+\frac{21}{\sqrt{21}}g_{2n+1}\ ,\quad n=0,1,2,3,\ldots$$

$$14y_{2n+1}-10=23f_{2n+1}+\frac{105}{\sqrt{21}}g_{2n+1}\ ,\quad n=0,1,2,3,\ldots$$

where

$$f_{2n+1}=\left[\left(55+12\sqrt{21}\right)^{2n+1}+\left(55-12\sqrt{21}\right)^{2n+1}\right]$$

$$g_{2n+1}=\left[\left(55+12\sqrt{21}\right)^{2n+1}-\left(55-12\sqrt{21}\right)^{2n+1}\right]$$

PROPERTIES:

- $7(x_{2n+3}-12649x_{2n+1})+139128\equiv 0(\bmod 2640)$
- $7y_{2n+3}+18480x_{2n+1}-88543y_{2n+1}+34200=0$
- $7x_{2n+3}+3857x_{2n+1}-18480y_{2n+1}+7128=0$
- $406560y_{2n+1}-84854x_{2n+1}-154x_{2n+3}$ is a perfect square
- $-399y_{2n+3}-1053360x_{2n+1}+5046951y_{2n+1}$ is a Nasty number

NOTE:

Instead of (18), if we consider the linear transformations

$Y=p-7q$ and $\alpha=p-3q$

then, the corresponding two sets (setV, setVI) of integer solutions to (1) are obtained as

Set V:

$$7x_{n+1}-2=-f_{n+1}\ ,\quad n=1,3,5,\ldots$$

$$14y_{n+1}-10=-5f_{n+1}-\frac{21}{\sqrt{21}}g_{n+1}\ ,\quad n=1,3,5,\ldots$$

Set VI:

$$14x_{n+1}-4=-23f_{n+1}-\frac{105}{\sqrt{21}}g_{n+1}\ ,\quad n=1,3,5,\ldots$$

$$14y_{n+1}-10=-5f_{n+1}-\frac{21}{\sqrt{21}}g_{n+1}\ ,\quad n=1,3,5,\ldots$$

where

$$f_{n+1}=\left[\left(55+12\sqrt{21}\right)^{n+1}+\left(55-12\sqrt{21}\right)^{n+1}\right]$$

$$g_{n+1} = \left[\left(55+12\sqrt{21}\right)^{n+1} - \left(55-12\sqrt{21}\right)^{n+1} \right]$$

CONCLUSION:

As the binary quadratic equations are rich in variety, one may consider other choices of hyperbolas and search for their non-trivial distinct integral solutions along with the corresponding properties.

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REFERENCES:

1. Dickson LE; History of Theory of numbers,Vol.2, Chelsea publishing company, Newyork,1952.
2. Mordel LJ; Diophantine Equations, Academic press, Newyork,1969.
3. Andre weil, Number Theory: An approach through history: from hammurapi to legendre \ Andre weil: Boston (Birkahasuser boston), 1983.
4. Nigel p. Smart, the algorithmic Resolutions of Diophantine equations, Cambridge university press, 1999.
5. Smith DE, History of mathematics Vol.I and II, Dover publications, New York, 1953.
6. Gopalan MA, Vidyalakshmi S, Devibala S; On the Diophantine equation $3x^2 + xy = 14$. Acta Ciencia Indica, 2007; XXXIIIM(2):645-646.
7. Gopalan MA, Janaki G; Observations on $Y^2 = 3X^2 + 1$", Acta ciencia Indica, 2008; XXXIVM(2):693-695.
8. Gopalan MA, Vijayalakshmi R; Special Pythagorean triangles generated through the integral solutions of the equation $y^2 = (K^2 + 1)x^2 + 1$. Antarctica J.Math, 2010; 7(5):503-507.
9. Gopalan MA, Sivagami B; Observations on the integral solutions of $y^2 = 7x^2 + 1$. Antartica J.Math, 2010; 7(3):291-296.
10. Gopalan MA, Vijayalakshmi R; Observation on the integral solutions of $y^2 = 5x^2 + 1$. Impact J.Sci.Tech, 2010; 4(4):125-129.
11. Gopalan MA, Sangeetha G; A remarkable observation on $y^2 = 10x^2 + 1$. Impact J. Sci. Tech, 2010; 4(1):103-106.
12. Gopalan MA, Parvathy G; Integral points on the Hyperbola $x^2 + 4xy + y^2 - 2x - 10y + 24 = 0$. Antarctica J. Math, 2010; 7(2):149-155.
13. Gopalan MA, Palanikumar R; Observations on $y^2 = 12x^2 + 1$. Antarctica J. Math, 2011; 8(2):149-152.
14. Gopalan MA, Devibala S, Vijayalakshmi R; Integral points on the hyperbola $2x^2 - 3y^2 = 5$. American Journal of Applied Mathematics and Mathematical Sciences, 2012; 1(1):1-4.
15. Gopalan MA, Vidyalakshmi S, Usha Rani TR, Mallika S; Observations on $y^2 = 12x^2 - 3$. Bessel J.Math, 2012; 2(3):153-158.
16. Gopalan MA, Vidyalakshmi S, Sumathi G, Lakshmi K; Integral points on the Hyperbola $x^2 + 6xy + y^2 + 40x + 8y + 40 = 0$. Bessel J. Math, 2012; 2(3):159-164.
17. Gopalan MA, Geetha K; Observations on the Hyperbola $y^2 = 18x^2 + 1$. Retell, 2012; 13(1):81-83.
18. Gopalan MA, Sangeetha G, Manju Somanath; Integral points on the Hyperbola $(a+2)x^2 - ay^2 = 4a(k-1) + 2k^2$. Indian Journal of Science, 2012; 1(2):125-126.
19. Gopalan MA, Vidyalakshmi S, Kavitha A; Observations on the Hyperbola $ax^2 - (a+1)y^2 = 3a - 1$. Discovery, 2013; 4(10):22-24.
20. Gopalan MA , Vidyalakshmi S, Kavitha A; Integral points on the Hyperbola $x^2 - 6xy + y^2 + 4x = 0$", Sch.J.Eng.Tech., 2014; 2(1):14-18.