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Research Article

Integral Points On The Cone $Z^2 = 41X^2 + Y^2$

K. Meena[1], S.Vidhyalakshmi[2], S.Divya[3*], M.A.Gopalan[4]
[1]Former VC of Bharathidasan University, Trichy-620024, Tamilnadu, India.
[3*]M.Phil student, Department of Mathematics, Shrimati Indira Gandhi College, Trichy- 620002, Tamilnadu, India.
[2,4]Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620002, Tamilnadu, India.

***Corresponding author**
S.Divya
Email: divya.kalavathy@gmail.com

Abstract: The ternary quadratic equation $Z^2 = 41X^2 + Y^2$ representing a homogeneous cone is analyzed for its non-zero distinct integer solutions. Also, given an integer solution, three different triples of integers generating infinitely many integer solutions are exhibited.
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INTRODUCTION

The ternary quadratic Diophantine equations offer an unlimited field for research due to their variety [1-2]. M.A.Gopalan et al [3-8] have analyzed integer solution for special 3-dimensional surfaces. This communication concerns with yet another interesting ternary quadratic equation $Z^2 = 41X^2 + Y^2$ representing a cone for determining its infinitely many non-zero integral points.

METHOD OF ANALYSIS:

The ternary quadratic equation to be solved for its non-zero integer solutions is

$$Z^2 = 41X^2 + Y^2 \tag{1}$$

To start with, it is noted that (1) is satisfied by

$$(X, Y, Z) = (1,20,21), (2pq, 41p^2 - q^2, 41p^2 + q^2), (11,-40,81), (-17,-124,165), (0,-50,50).$$

However, we have other patterns of solutions which are illustrated below.

PATTERN: 1

(1) is written as the system of double equations given by

$$\left.\begin{aligned} Z + Y &= 41X^2 \\ Z - Y &= 1 \end{aligned}\right\} \tag{2}$$

Solving (2), we have

$$Y = \frac{41X^2 - 1}{2}, \quad Z = \frac{41X^2 + 1}{2}$$

Since our aim is to find integer solutions, note that Y and Z are integers provided X is odd. Thus the non-zero distinct integer solutions to (1) are obtained as

$$X = 2k \pm 1,$$

$$Y = 41(2k^2 \pm 2k) + 20,$$

$$Z = 41(2k^2 \pm 2k) + 21.$$

PATTERN: 2

Writing (1) as the system of equations given by

$$\left.\begin{aligned} Z+Y&=X^2\\ Z-Y&=41\end{aligned}\right\}\qquad(3)$$

Following the procedure presented in pattern: 1, the corresponding values of X, Y and Z obtained from (3) are

$X = 2k \pm 1,$

$Y = 2k^2 \pm 2k - 20,$

$Z = 2k^2 \pm 2k + 21.$

PATTERN: 3

(1)Can be written as $Z^2 - 41X^2 = Y^2 * 1$ (4)

Assume $Y = Y(a,b) = a^2 - 41b^2; a,b > 0$ (5)

Write 1 as

$$1 = \frac{(25+3\sqrt{41})(25-3\sqrt{41})}{256} \qquad (6)$$

Substituting (5) and (6) in (4) and employing the method of factorization, define

$$(Z + \sqrt{41}X) = (a + \sqrt{41}b)^2 \frac{(25+3\sqrt{41})}{16}$$

Equating rational and irrational factors, we get

$$\left.\begin{aligned} Z = Z(a,b) &= \frac{1}{16}[25a^2 + 1025b^2 + 246ab]\\ X = X(a,b) &= \frac{1}{16}[3a^2 + 123b^2 + 50ab]\end{aligned}\right\}\qquad(7)$$

As our interest is on finding integer solutions, we choose a and b suitably so that the values of X, Y and Z are in integers. Replacing a by 4A and b by 4B in (5) and (7), the corresponding integer solutions to (1) in two parameters are given by

$$\left.\begin{aligned} X = X(A,B) &= 3A^2 + 123B^2 + 50AB\\ Y = Y(A,B) &= 16A^2 - 656B^2\\ Z = Z(A,B) &= 25A^2 + 1025B^2 + 246AB\end{aligned}\right\}\qquad(8)$$

PATTERN: 4

Instead of (6), write 1 as

$$1 = \frac{(21+\sqrt{41})(21-\sqrt{41})}{400} \qquad (9)$$

Following the procedure presented in pattern: 3 and replacing a by 20A and b by 20B, the corresponding integer solutions of (1) in two parameters are found to be

$$\left.\begin{aligned} X = X(A,B) &= 20A^2 + 820B^2 + 840AB\\ Y = Y(A,B) &= 400A^2 - 16400B^2\\ Z = Z(A,B) &= 420A^2 + 17220B^2 + 1640AB\end{aligned}\right\}\qquad(10)$$

PATTERN: 5

(1) Can be written as $41X^2 + Y^2 = Z^2 * 1$ (11)

Assume $Z = Z(a,b) = a^2 + 41b^2; a,b > 0$ (12)

Write 1 as

$$1 = \frac{(16+i3\sqrt{41})(16-i3\sqrt{41})}{625} \qquad (13)$$

Following the procedure presented in pattern: 3 and replacing a by 5A and b by 5B, the corresponding integer solutions of (1) in two parameters are found to be

$$\left.\begin{aligned} X &= X(A,B) = 3A^2 - 123B^2 + 32AB \\ Y &= Y(A,B) = 16A^2 - 656B^2 - 246AB \\ Z &= Z(A,B) = 25A^2 + 1025B^2 \end{aligned}\right\} \quad (14)$$

PATTERN: 6

Instead of (13) write (1) as

$$1 = \frac{(8 + i5\sqrt{41})(8 - i5\sqrt{41})}{1089} \quad (15)$$

Following the procedure presented in pattern: 3 and replacing a by 33A and b by 33B, the corresponding integer solutions of (1) in two parameters are found to be

$$\left.\begin{aligned} X &= X(A,B) = 165A^2 - 6765B^2 + 528AB \\ Y &= Y(A,B) = 264A^2 - 10824B^2 - 13530AB \\ Z &= Z(A,B) = 1089A^2 + 44649B^2 \end{aligned}\right\} \quad (16)$$

GENERATION OF SOLUTIONS:

Let (X_0, Y_0, Z_0) be any given integer solution of (1).

Let $X_1 = 7X_0, Y_1 = 7Y_0 + 4h, Z_1 = 7Z_0 + 3h, h \neq 0$ (17)

be the second solution of (1) then, from (1) and (17), we have

$h = 6Z_0 - 8Y_0$ and thus, we have

$$Y_1 = 24Z_0 - 25Y_0$$

$$Z_1 = 25Z_0 - 24Y_0$$

which is written in the matrix form as

$$\begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix} = A \begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix} \quad (18)$$

Where $A = \begin{pmatrix} -25 & 24 \\ -24 & 25 \end{pmatrix}$

Repeating the above process, one obtains

$$\begin{pmatrix} Y_n \\ Z_n \end{pmatrix} = \frac{1}{14} \begin{bmatrix} -18\alpha^n + 32\beta^n & 24(\alpha^n - \beta^n) \\ -24(\alpha^n - \beta^n) & 32\alpha^n - 18\beta^n \end{bmatrix} \begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix}$$

Hence knowing a solution of (1), the formula for generating a sequence of integer solutions to (1) is represented as follows:

$$X_n = 7^n X_0$$

$$Y_n = \frac{1}{14}\left[(-18\alpha^n + 32\beta^n)Y_0 + 24(\alpha^n - \beta^n)Z_0\right]$$

$$Z_n = \frac{1}{14}\left[-24(\alpha^n - \beta^n)Y_0 + (32\alpha^n - 18\beta^n)Z_0\right]$$

In which $\alpha = 7, \beta = -7$

However, in addition to (17), we have two more choices of linear transformations to generate sequences of integer solutions to (1) which are exhibited below:

Linear transformation 1:

$$X_1 = 5X_0 + h, Y_1 = 5Y_0, Z_1 = 5Z_0 + 6h$$

The corresponding general solution of (1) is given by

$$X_n = \frac{1}{10}\left[(-82\alpha^n + 72\beta^n)X_0 + 12(\alpha^n - \beta^n)Z_0\right]$$

$$Y_n = 5^n Y_0$$

$$Z_n = \frac{1}{10}\left[-492(\alpha^n - \beta^n)X_0 + (72\alpha^n - 82\beta^n)Z_0\right]$$

In which $\alpha = 5, \beta = -5$

Linear transformation 2:

$$X_1 = 42X_0 + h, Y_1 = 42Y_0 + h, Z_1 = 42Z_0$$

For this choice, we have

$$X_n = \frac{1}{84}\left[(2\alpha^n + 82\beta^n)X_0 - 2(\alpha^n - \beta^n)Y_0\right]$$

$$Y_n = \frac{1}{84}\left[-82(\alpha^n - \beta^n)X_0 + (82\alpha^n + 2\beta^n)Y_0\right]_{\mathbf{s}}$$

$$Z_n = 42^n Z_0$$

In which $\alpha = 42, \beta = -42$

CONCLUSION

In this paper, we have presented different patterns of integer solutions to the ternary quadratic Diophantine equation and three different formulas to generate sequences of integer solutions being given an solution to the cone under consideration.

To conclude, one may search for other patterns of solution and their corresponding properties.

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