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Purity in Module Theory and Its Relatives: A Survey of Foundations, Recent Advances and Open Problems

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Abstract | **Review Article**

Cohn introduced purity for modules as a tensor-theoretic weakening of splitting. Since then, it has developed into a family of related notions, used in classical ring theory as well as in exact, Grothendieck and triangulated categories. In this survey we organise the family around one question: which short exact sequences should be treated as almost split, and what do the corresponding relative injective, projective and flat modules say about the ring? We first set up a common language of proper classes defined by test modules, and place cyclic, relative-divisibility, neat, simple-module and torsion-theoretic purities, with their copure duals, in it. We then report on work from the last four years: purities attached to multiplicative subsets and their uniform versions, strong purity and S-phantom morphisms, pure-direct-injective and pure-direct-projective modules, Mittag-Leffler and definability methods, and purity in exact and triangulated categories. Primeness of submodules and of ideals in near-rings is discussed as well, because several recent papers treat both topics. The papers of the second author on torsion purity, T-purity, divisibility, cyclic purity and σ-purity are compared with this literature. Along the way we prove a few short results that we could not find in print: a relative form of Chase's theorem on products of flat modules, a torsion-theoretic form of Warfield's theorem on Prüfer domains, the existence of envelopes for τ-injective modules and covers for τ-divisible modules, and a reduction of relative pure semisimplicity to the absolute case. A count of the one hundred references by theme and period shows where the activity is concentrated. We end with twelve open problems, each with its motivation and a suggested approach.

Keywords: Pure Submodule, Relative Purity, Proper Class, Torsion Theory, Absolutely Pure Module, Pure-Injective Module, Divisible Module, Copurity, Multiplicative Subset, Near-Ring.

Mathematics Subject Classification (2020): Primary 16D40, 16D50; Secondary 13C11, 16D80, 16S90, 18G25, 16Y30.

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1. INTRODUCTION

A short exact sequence that splits is as well behaved as one could wish, but over most rings very few sequences split, and a theory that only sees split sequences says little. Purity is the standard way of relaxing the requirement. Informally, a submodule N of a left R-module M is pure if any finite system of linear equations with constants in N which can be solved in M can already be solved in N. The same condition can be phrased with tensor products: $N \hookrightarrow M$ stays injective after tensoring with any right module. Cohn [20], introduced the tensor version in his work on free products of rings. Warfield [88], studied the equational version in depth, together with algebraically compact (that is, pure-injective) modules. A little earlier Stenström [85], and then Fieldhouse [34], had already noticed that the definition really depends on a choice of "test" modules: a sequence is pure for a class of test modules if no module in the class can tell that the sequence does not split.

Once this is seen, purity stops being one notion and becomes a way of manufacturing notions. Each choice of test class produces a purity, and with it a class of relatively injective, projective and flat modules. Sklyarenko [84], organised the resulting proper classes in a long survey, and the books of Enochs and Jenda [30], Göbel and Trlifaj [37], and Prest [77], developed approximation theory and model theory on this foundation. New examples keep appearing. Purities

defined by simple, cyclic or cyclically presented modules, or by a hereditary torsion theory, each have a literature of their own. Since about 2020 further variants have come from three other directions: multiplicative subsets of commutative rings [32-93], exact structures on additive categories [75-82], and compactly generated triangulated categories [14, 15].

The second author and his co-workers have written on several of these topics over the past twenty-five years. The papers concern cyclic purity and its co-cyclic dual [72]; ideal purity and absolute purity [59], torsion purity and M-purity [70, 71], purity relative to a single cyclic module [64], σ-purity, σ-regularity, σ-projectivity, σ-semisimplicity and local σ-projectivity [65-69], T-purity and T-copurity [67], and divisibility and co-divisibility [68]. A second group of papers deals with prime ideals of near-rings and with submodule conditions over near-ring-like structures [60–63]. Most of these were published in journals that are not widely read outside India, and they have rarely been compared with the current literature on the same questions. Making that comparison is one purpose of this article.

We have three aims. The first is to give a common vocabulary in which different purities can be compared (Sections 2 and 3). The second is to report on recent work, mostly from 2022 to 2026. Sections 4 to 9 take the main threads in turn: torsion-theoretic purity and divisibility, regularity-type conditions, purity relative to a multiplicative subset, pure-injective and pure-projective modules, purity in categories other than module categories, and primeness in modules and near-rings. The third aim is to point out what is still open. Section 10 looks at the hundred references quantitatively, and Section 11 lists twelve open problems, each with a motivation and a suggested line of attack.

While collecting the material we noticed that a few natural questions have short answers which we could not find in print. They are recorded, with proofs, as Propositions 3.1, 4.3, 4.4 and 7.1 (and Corollary 7.2). None of them is deep, but each one either settles a question one would be tempted to ask or moves it to a sharper form, and the problems in Section 11 are stated with them in mind.

Elsewhere we give statements in a form that can be quoted, but proofs are only sketched, and only when a few lines explain why a result is true. Full arguments are in the cited papers.

Conventions

Rings are associative with 1 and modules are unital. "Module" means left module unless we say otherwise; R-Mod is the category of left R-modules and mod$R$ its full subcategory of finitely presented modules. The character module of a left module M is the right module $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. When R is commutative, S is always a multiplicatively closed subset of R with $1 \in S$.

2. Purity: Definitions and Basic Facts

2.1. Equivalent Descriptions

Let

$$\mathbb{E}: \quad 0 \to N \xrightarrow{f} M \xrightarrow{g} L \to 0$$

Be a short exact sequence of left R-modules. The four conditions below are equivalent [20, 34, 88]. If they hold, $\mathbb{E}$ is *pure exact*, f is a *pure monomorphism*, g a *pure epimorphism*, and $f(N)$ a *pure submodule* of M.

(P1) $X \otimes_R f$ is injective for every right R-module X.

(P2) $\mathrm{Hom}_R(F, M) \to \mathrm{Hom}_R(F, L)$ is onto for every finitely presented left module F.

(P3) Whenever a finite system

$$\sum_{j=1}^{n} r_{ij}\, x_j = a_i \qquad (1 \le i \le m), \qquad r_{ij} \in R,\ a_i \in f(N),$$

has a solution $(x_1, \dots, x_n) \in M^n$, it has a solution in $f(N)^n$.

(P4) The character sequence $0 \to L^+ \to M^+ \to N^+ \to 0$ splits.

In (P1) it is enough to let X run over finitely presented right modules, since every module is a direct limit of finitely presented ones and $\otimes$ commutes with direct limits. Going from (P2) to (P3) is mostly a matter of translation. A finite system with constants in $f(N)$ amounts to a homomorphism from a finitely presented module, and the system is solvable in M precisely when that homomorphism factors in the right way. Condition (P4) is the reason character modules are so useful here: M is flat if and only if M^+ is injective, and character modules are always pure-injective (Definition 2.2).

Over a principal ideal domain, (P3) reduces to the condition $rN = N \cap rM$ for every $r \in R$, which is how purity has been defined for abelian groups since Prüfer. Over other rings this "relative divisibility" condition is weaker than purity. Where exactly the two separate is one of the first non-trivial questions in the subject (Section 3.2).

2.2. Purity Relative to a Class of Modules

The step that gives the theory its flexibility is the following.

Definition 2.1: Let $\mathcal{S}$ be a class of right R-modules and $\mathcal{T}$ a class of left R-modules. A short exact sequence $\mathbb{E}$ of left modules is

(a) *$\mathcal{S}$-flatly pure* (or $\mathcal{S}$-pure in the tensor sense) if $X \otimes_R \mathbb{E}$ is exact for all $X \in \mathcal{S}$;

(b) *$\mathcal{T}$-projectively pure* if $\mathrm{Hom}_R(T, \mathbb{E})$ is exact for all $T \in \mathcal{T}$.

In both cases one gets a *proper class* in the sense of Buchsbaum and of Mishina and Skornyakov. Such a class contains the split sequences; it is closed under isomorphism, pullbacks and pushouts; and its monomorphisms, and also its epimorphisms, are closed under composition [84]. In the language of exact categories a proper class is just an exact structure on R-Mod contained in the abelian one [24-82]. Two remarks help with the bookkeeping. A larger $\mathcal{S}$ or $\mathcal{T}$ gives fewer pure sequences. And Cohn purity is the strongest purity of tensor type, since a sequence satisfying (P1) is $\mathcal{S}$-flatly pure for every $\mathcal{S}$.

Definition 2.2

Let $\mathcal{P}$ be a proper class. A module Q is $\mathcal{P}$-*injective* if $\mathrm{Hom}_R(\mathbb{E}, Q)$ is exact for all $\mathbb{E} \in \mathcal{P}$, and $\mathcal{P}$-*projective* is defined dually. A right module X is $\mathcal{P}$-*flat* if $X \otimes_R \mathbb{E}$ is exact for all $\mathbb{E} \in \mathcal{P}$. A module D is *absolutely* $\mathcal{P}$-*pure* if every short exact sequence starting at D belongs to $\mathcal{P}$.

For Cohn purity these four classes are the pure-injective, pure-projective, flat and absolutely pure modules. Absolutely pure modules go back to Maddox [50]. Megibben [55], showed that they are exactly the modules D with $\mathrm{Ext}^1_R(F, D) = 0$ for all finitely presented F; in this description they are usually called FP-injective. We recall the main structure theorems, because almost every other purity has been given analogues of them.

Theorem 2.3 [55, 86, 88]. Let R be a ring.

1. R is von Neumann regular if and only if every left R-module is absolutely pure.
2. R is left Noetherian if and only if every absolutely pure left R-module is injective.
3. R is left coherent if and only if the absolutely pure left modules are closed under direct limits.
4. A module is pure-projective if and only if it is a direct summand of a direct sum of finitely presented modules, and every module has a pure-injective envelope.

Fieldhouse [35], added a companion to (1). Call a module *regular* if all its submodules are pure; then R is von Neumann regular exactly when every left module is regular. Theorem 2.3 also shows the typical form of a result about a purity: some homological property of the modules attached to the purity holds if and only if the ring satisfies a finiteness or regularity condition. Most of the results in Sections 3 to 7 are of this form.

2.3. Torsion Theories

A *torsion theory* on R-Mod is a pair $\tau = (\mathcal{T}, \mathcal{F})$ of classes of modules such that $\mathrm{Hom}_R(T, F) = 0$ for $T \in \mathcal{T}$ and $F \in \mathcal{F}$, and each class is maximal with this property. It is *hereditary* if $\mathcal{T}$ is closed under submodules. In that case every module M has a largest torsion submodule $\tau(M)$, and $M/\tau(M)$ lies in $\mathcal{F}$. We say that $N \leq M$ is τ-*dense* if $M/N \in \mathcal{T}$, and τ-*pure* (other names are τ-closed and saturated) if $M/N \in \mathcal{F}$. A module D is τ-*injective* (Lambek says τ-divisible) if every map into D from a τ-dense submodule of a module B extends to B. The torsion theory is determined by its *Gabriel filter* $\mathfrak{F}$, the set of left ideals I with $R/I \in \mathcal{T}$, and τ is *of finite type* if $\mathfrak{F}$ has a basis of finitely generated left ideals.

Lemma 2.4: For a hereditary torsion theory τ, a module D is τ-injective if and only if $\mathrm{Ext}^1_R(T, D) = 0$ for all $T \in \mathcal{T}$.

Sketch

Let $A \leq B$ with $T = B/A$ torsion. In the exact sequence $\mathrm{Hom}(B, D) \to \mathrm{Hom}(A, D) \to \mathrm{Ext}^1(T, D)$ the vanishing of the last term gives the extension property. In the other direction write $T = P/K$ with P projective. Then K is τ-dense in P, and if every map $K \to D$ extends to P then $\mathrm{Ext}^1(T, D) = 0$. ▫

Two torsion theories will come up several times. For the *Goldie torsion theory* the torsion modules are the essential extensions of singular modules. A submodule $N \leq M$ is then pure exactly when M/N is nonsingular. Such submodules are always closed (complement) submodules, and in a nonsingular module the converse holds as well. For the *Dickson torsion theory* the torsion modules are the semiartinian ones, in which every nonzero quotient has nonzero socle. Crivei [23], proved that the absolutely neat modules are precisely the injectives relative to this theory; this links the torsion-theoretic approach with the purities defined by simple modules (Section 3.3).

3. Purities Generated By Classes of Test Modules

3.1. An Overview

Most purities in current use come from Definition 2.1 with a class of cyclic, simple or finitely presented test modules. Table 1 collects the ones met most often, and Figure 1 compares the corresponding proper classes. An arrow from one purity to another means that every sequence which is pure in the first sense is pure in the second; so arrows point from stronger to weaker purities.

Nearly all the arrows in Figure 1 come from comparing test classes. Cyclically presented modules R/rR are cyclic, so cyclic purity implies RD-purity; finitely n-presented modules are finitely presented, so purity implies n-purity. Cohn purity implies every purity of tensor type, s-purity among them. It implies neatness only when simple modules are finitely presented, because (P2) says nothing about $\mathrm{Hom}(F, -)$ for F not finitely presented. There is no arrow between τ-purity and the rest. The definition of τ-purity is a condition on the quotient rather than a test class, and whether it is stronger or weaker than Cohn purity depends on τ (Section 4.1).

Table 1: Some purities defined by a class of test modules. The third column gives the condition on a submodule $N \leq M$, and the last column gives one structural result of the kind described after Theorem 2.3.

Purity	Test class	Condition on $N \leq M$	A representative result
Cohn purity	all right modules (equivalently $\mathrm{mod}R^{\mathrm{op}}$)	$X \otimes N \to X \otimes M$ injective for all X	R is von Neumann regular iff every submodule of every module is pure [35]
RD-purity	R/rR, $r \in R$	$N \cap rM = rN$ for all r	a commutative domain is Prüfer iff RD-purity equals purity [88]
Cyclic purity	R/I, I a right ideal	$N \cap IM = IN$ for all I	a commutative ring is pure-semisimple iff every module is cyclically pure injective [26]
s-purity (coneatness)	simple right modules S	$S \otimes N \to S \otimes M$ injective	a commutative ring is hereditary Noetherian iff it is nonsingular and absolutely s-pure modules are injective [18]
Neatness	simple left modules S	$\mathrm{Hom}(S, M) \to \mathrm{Hom}(S, M/N)$ onto	absolutely neat modules are the injectives for the Dickson torsion theory [23]
n-purity	finitely n-presented modules	$\mathrm{Hom}(F, -)$ exact on the sequence	characterises strong n-coherence [38]
τ-purity	a hereditary torsion theory τ	M/N is τ-torsion-free	Section 4
S-purity (R commutative)	a multiplicative set S	$s(N \cap IM) \subseteq IN$ for some $s \in S$, all ideals I	Section 6

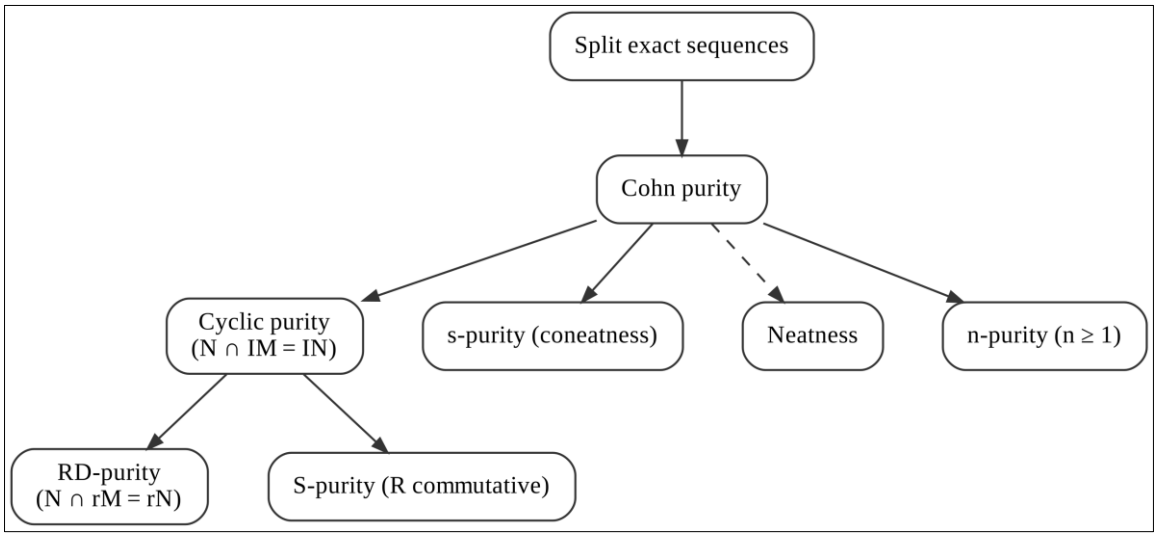


Figure 1: Comparison of some proper classes of short exact sequences. An arrow $\mathcal{P}_1 \to \mathcal{P}_2$ means $\mathcal{P}_1 \subseteq \mathcal{P}_2$. The dashed arrow holds when every simple left module is finitely presented, for instance over a left Noetherian ring; the arrow to S-purity concerns commutative rings

3.2. Relative Divisibility and Cyclic Purity

Many later papers start from the comparison of RD-purity with purity. By Warfield's theorem [88], the two agree over a commutative domain if and only if the domain is Prüfer; in other words, whether two purities coincide is an arithmetic property of the ring. Puninski, Prest and Rothmaler [79], took this as a programme and studied the rings that are characterised by coincidences between purities of various strengths. Couchot [21], worked out the RD-analogues of flat and FP-injective modules (RD-flat and RD-coflat modules). One of his results concerns commutative rings over which RD-injective modules are always Σ-RD-injective: such a ring is a product of a pure-semisimple ring and a finite ring. Moradzadeh-Dehkordi and Couchot [57] then looked at simple modules and found when they are RD-flat or RD-injective.

Work on cyclic purity has gone in two directions. With the tensor definition, $N \leq M$ is cyclically pure if $N \cap IM = IN$ for all right ideals I. Divaani-Aazar, Esmkhani and Tousi [25], characterised the cyclically pure injective modules as the direct summands of modules $\mathrm{Hom}_R(L, E)$ with E injective and L a direct sum of finitely presented cyclic modules. In a second paper [26] they proved that a commutative domain is Prüfer as soon as two of the three classes of pure-injective, cyclically pure injective and RD-injective modules coincide, and conversely. With the Hom definition, Behboodi, Ghorbani, Moradzadeh-Dehkordi and Shojaee [12], considered C-pure projective modules,

i.e. modules projective with respect to sequences on which $\mathrm{Hom}(C,-)$ is exact for all cyclic C. They showed that every C-pure projective left module is pure-projective if and only if R is left Noetherian. Over a left hereditary Noetherian ring they also described the C-pure projective modules completely: they are the modules $N \oplus P$ with P projective and N a direct sum of cyclic modules.

Two papers of the second author fit here: one on cyclic purity and co-cyclic copurity in module categories [72], and one on purity relative to a cyclic module [64]. If the test class consists of a single cyclic module $C = R/I$, the tensor condition becomes $N \cap IM = IN$ for this one right ideal I, and the proper class obtained sits between Cohn purity and the split sequences in a way that depends on I rather subtly. One-module purities are easy to compute with, and every purity defined by a class is the intersection of the purities defined by its members, so they are a convenient place to test conjectures.

3.3. Purities Generated by Simple Modules

In an abelian group, a subgroup $A \leq B$ is neat if $pA = A \cap pB$ for every prime p. For modules one uses simple modules instead of primes. A submodule $N \leq M$ is *neat* if $\mathrm{Hom}_R(S,-)$ is exact on $0 \to N \to M \to M/N \to 0$ for every simple module S, and *coneat* (or *s-pure*) if $S \otimes_R -$ is exact on it for every simple right module S. Crivei [23] extended a result of Fuchs by showing that the two notions agree over a commutative ring whose maximal ideals are all principal; as mentioned in Section 2.3, he also identified the absolutely neat modules with the Dickson-injective ones.

Büyükaşık and Durğun [18], then studied absolutely s-pure modules and neat-flat modules. A module M is neat-flat if every epimorphism onto M has neat kernel. One of their results characterises right Kasch rings: R is right Kasch if and only if the injective hulls of simple right modules are neat-flat, and if and only if every absolutely pure right module is neat-flat. Alagöz and Büyükaşık [3], defined max-flat modules by the vanishing of $\mathrm{Tor}_1^R(A, R/I)$ for all maximal left ideals I, together with the right orthogonal class of max-cotorsion modules. Every module has a max-flat cover and a max-cotorsion envelope. For a left perfect, right max-injective ring they also showed that being quasi-Frobenius is the same as having finitely generated maximal right ideals. Alagöz, Alizade, Büyükaşık and Sağbaş [2], went one step further and tested only against simple right ideals. They defined weakly neat-flat and weakly absolutely s-pure modules and showed that these coincide with the unweakened notions precisely over right Kasch rings; one of the two equivalences needs a min-coherence assumption.

Durğun [28], compared four proper classes, those generated by closed, neat, pure and C-pure submodules. He determined the rings over which every finitely generated (or every cyclic, or every injective) module is closed-projective. Pure submodules have also been used to modify lattice-theoretic notions. Ahmed [1], tested essentiality only against pure submodules, which leads to P-essential submodules and P-uniform modules; Ibrahim and Al-Mothafar [45], studied pure essential submodules, pure relative complements and pure uniform modules. In both papers a purity is fixed first, and one then asks how much of the theory of essential extensions still holds when only pure submodules are taken into account. As one would expect, the answer is different for different purities.

3.4. Higher Finite Presentation and Products of Relatively Flat Modules

A module is *finitely n-presented* if it has a projective resolution whose first $n+1$ terms are finitely generated; for $n = 2$ we also say FP_2. Using finitely n-presented modules in place of finitely presented ones in (P2) gives *n-purity*. The absolute objects are now the FP_n-injective modules, and the matching ring condition is n-coherence. Gubitosi and Parra [38], used n-pure exact sequences to show that FP_n-injective and FP_n-flat modules are dual to each other. They characterised strong n-coherent rings and introduced an FP_n-projective global dimension which measures how far a ring is from being Noetherian. Parra [73], restricted the test class further to finitely n-presented modules of projective dimension at most d. This produces relative injective, projective, flat and cotorsion classes and new characterisations of the (n,d)-coherent rings of Mao and Ding; when d is large one gets back Costa's n-coherence. In a related direction Lu [49], described the FP-injective objects in the category of N-complexes as the N-exact complexes with FP-injective cycle modules, and used this over coherent rings.

The question of when relatively flat modules are closed under direct products comes up for each of these purities, and it was answered separately for each: for flat modules by Chase, for RD-flat modules [21], for max-flat modules [3], and for FP_n-flat modules [38]. Chase's argument in fact works for any set of finitely presented test modules. For a set $\mathcal{S}$ of finitely presented right modules, call a left module C *$\mathcal{S}$-flat* if $\mathrm{Tor}_1^R(X,C) = 0$ for all $X \in \mathcal{S}$. If $0 \to K \to F \to C \to 0$ is exact with F free, then $X \otimes K \to X \otimes F$ is injective exactly when $\mathrm{Tor}_1^R(X,C) = 0$. So the $\mathcal{S}$-flat modules are the modules C for which every short exact sequence ending in C is $\mathcal{S}$-flatly pure, that is, the relatively flat modules of this proper class.

Proposition 3.1: Let $\mathcal{S}$ be a set of finitely presented right R-modules. The following are equivalent.

1. Direct products of $\mathcal{S}$-flat left modules are $\mathcal{S}$-flat.
2. R^I is $\mathcal{S}$-flat for every set I.
3. Every $X \in \mathcal{S}$ is FP_2.

Proof. (1) implies (2) because R is $\mathcal{S}$-flat. Assume (2) and let $X \in \mathcal{S}$. Choose an exact sequence $0 \to K \to P \to X \to 0$ with P finitely generated free; K is finitely generated. Since $\mathrm{Tor}_1(X, R^I) = 0$, the sequence $0 \to K \otimes R^I \to P \otimes R^I \to X \otimes R^I \to 0$ is exact. It maps to $0 \to K^I \to P^I \to X^I \to 0$ by the canonical maps, and these are bijective for P and X because both are finitely presented. By the five lemma $K \otimes R^I \to K^I$ is bijective for every I, so the finitely generated module K is finitely presented and X is FP_2. Finally, if X is FP_2 then $\mathrm{Tor}_1(X, -)$ commutes with direct products, which gives (1). ▫

Specialising $\mathcal{S}$ recovers the known cases. All finitely presented modules give Chase's theorem (right coherence). The modules R/rR give the condition that every element has a finitely generated right annihilator. Finitely n-presented modules give n-coherence in the sense of Costa and of Chen and Ding. And the modules $R/\mathfrak{m}$, with $\mathfrak{m}$ a finitely generated maximal right ideal, give the condition that such $\mathfrak{m}$ are finitely presented. Proposition 3.1 does not cover test modules that fail to be finitely presented, such as arbitrary simple modules. That case is still open and appears as Problem 11.6.

4. Torsion-Theoretic Purity, Divisibility and Copurity

4.1. Purity Relative to a Torsion Theory

Let τ be a hereditary torsion theory. In any module M the τ-pure submodules form a lattice, closed under arbitrary intersections, and each submodule N has a smallest τ-pure submodule containing it: the preimage in M of $\tau(M/N)$. To see how τ-purity relates to the purities of Section 3, the simplest case is a commutative domain R with its classical torsion theory, in which a module is torsion when every element has a nonzero annihilator.

Proposition 4.1: Let R be a commutative domain with the classical torsion theory τ, and let $N \leq M$.

1. If N is τ-pure in M, then N is RD-pure in M.
2. If M is torsion-free, then N is τ-pure in M if and only if it is RD-pure in M.
3. If R is a Prüfer domain and M is torsion-free, then τ-purity, RD-purity and Cohn purity coincide for submodules of M.

Sketch. (1) Let $x = rm \in N \cap rM$ with $r \neq 0$. Then $r(m + N) = 0$ in M/N, which is torsion-free, so $m \in N$ and $x \in rN$. (2) Suppose N is RD-pure and $r(m + N) = 0$ with $r \neq 0$. Then $rm \in N \cap rM = rN$, say $rm = rn$, and $m = n \in N$ because M is torsion-free. (3) follows from (2) and Warfield's theorem (Table 1). ▫

The proposition is easy, but it shows where the difficulty is. On modules with torsion, τ-purity is strictly stronger than RD-purity: 0 is RD-pure in $\mathbb{Z}/2\mathbb{Z}$ but is not τ-pure. The same example shows that τ-pure sequences do not form a proper class in the sense of Section 2.2, because a split sequence $0 \to A \to A \oplus T \to T \to 0$ with $T \neq 0$ torsion is not τ-pure. On torsion-free modules things are much better, and Proposition 4.1 has a version for every hereditary torsion theory. Recall from Section 2.3 that an element is torsion exactly when its annihilator lies in the Gabriel filter $\mathfrak{F}$.

Proposition 4.2: Let τ be a hereditary torsion theory with Gabriel filter $\mathfrak{F}$ and let M be τ-torsion-free. For $N \leq M$ the following are equivalent:

1. N is τ-pure in M;
2. $\mathrm{Hom}_R(R/I, -)$ is exact on $0 \to N \to M \to M/N \to 0$ for every $I \in \mathfrak{F}$.

If in addition τ is of finite type, every pure submodule of M is τ-pure.

Sketch. A map $R/I \to M/N$ is given by an element $\bar{x}$ with $I\bar{x} = 0$, and lifting it to M means finding a preimage of $\bar{x}$ that is killed by I. As M is torsion-free and $I \in \mathfrak{F}$, the only element of M killed by I is 0. So (2) says that no nonzero element of M/N is annihilated by a member of $\mathfrak{F}$, which is (1). When $\mathfrak{F}$ has a basis of finitely generated ideals, the relevant modules R/I are finitely presented, and the last assertion follows from (P2). ▫

Thus on torsion-free modules τ-purity is a projectively generated purity in disguise. One can go further and ask when it coincides with Cohn purity there. In the classical case of a domain the answer is Warfield's: exactly for Prüfer domains. The general answer turns out to be equally short. In the next statement $\bar{R} = R/\tau(R)$, and $\bar{\tau}$ is the torsion theory induced on $\bar{R}$-modules.

Proposition 4.3: Let τ be a hereditary torsion theory of finite type. The following are equivalent.

1. For every τ-torsion-free module M, the τ-pure submodules of M are exactly its pure submodules.
2. Every $\bar{\tau}$-torsion-free $\bar{R}$-module is flat over $\bar{R}$.

Proof

First, $\tau(R)$ is a two-sided ideal (it is invariant under the right multiplications, which are the endomorphisms of ${}_R R$), and $\tau(R)M = 0$ whenever M is torsion-free. Indeed, for $r \in \tau(R)$ and $x \in M$ the left annihilator of rx contains that of r, which lies in $\mathfrak{F}$, so rx is torsion and hence 0. For $\bar{R}$-modules, purity over R and over $\bar{R}$ are the same thing, because $A \otimes_R N \cong \big(A/A\tau(R)\big) \otimes_{\bar{R}} N$. We may therefore suppose $\tau(R) = 0$; in particular free modules are now torsion-free. Assume (1) and let C be torsion-free. Write $C \cong F/K$ with F free. Then K is τ-pure in F, hence pure by (1), and C is flat. Conversely, assume (2), let M be torsion-free and $N \leq M$ be τ-pure. Then M/N is torsion-free, hence flat, and so N is pure in M. The reverse inclusion is the last part of Proposition 4.2. ▫

For a commutative domain and the classical theory, condition (2) says that torsion-free modules are

flat, i.e. that R is Prüfer, so Warfield's theorem is a special case. In general the proposition replaces the comparison of two purities by a property of a single class of modules, and the interesting question becomes which rings have it for a given τ. This is Problem 11.2.

The papers of Pandey and Pathak on torsion purity in rings and modules [70], and on M-purity and torsion purity [71], as well as the second author's paper on T-purity and T-copurity [67], study exactly this kind of situation: a purity defined by a torsion class is compared with purities defined by test modules, and properties of the associated injective and flat modules are deduced. Two recent lines of work are relevant. Arellano, Castro and Ríos [8], worked inside $\sigma[M]$ with a hereditary torsion theory τ. They took the intersection of all τ-pure prime submodules (the τ-pure prime radical) and showed that it is τ-nilpotent under three assumptions: M is a finitely generated progenerator of $\sigma[M]$, τ is FIS-invariant, and M has Krull dimension relative to τ. Mazari-Armida [54], approached torsion modules from model theory. He treated torsion modules with pure embeddings as an abstract elementary class and, under the assumption that the torsion submodule is pure, characterised the pure-injective and Σ-pure-injective modules relative to this class; the results generalise the theorems of Gruson–Jensen and Zimmermann on Σ-pure-injectivity.

4.2. Divisibility

A module D over a commutative domain is divisible if $rD = D$ for every $r \neq 0$. Injective modules are divisible, and divisible modules are all injective exactly when the domain is Dedekind. For a hereditary torsion theory τ two generalisations are in use, and it is worth keeping them apart. Lambek's "τ-divisible" modules are what we called τ-injective in Section 2.3. We will call D *τ-divisible* in the narrower sense if $ID = D$ for every $I \in \mathfrak{F}$; for the classical torsion theory of a domain this is ordinary divisibility. Bland [16], introduced codivisible modules as the dual notion and developed both sides together.

By Lemma 2.4, τ-injectivity is the vanishing of $\mathrm{Ext}^1_R(\mathcal{T}, -)$. In other words, it is injectivity relative to the proper class of short exact sequences $0 \to A \to B \to C \to 0$ with C torsion. Statements about such modules are therefore statements about a proper class, which is one reason divisibility and purity so often appear side by side. Crivei and Keskin Tütüncü [24], carried this out in an exact category with two exact structures $\mathcal{E} \subseteq \mathcal{D}$. An object X is *$\mathcal{E}$-divisible* if every $\mathcal{D}$-sequence starting at X is in $\mathcal{E}$, and *$\mathcal{E}$-flat* if every $\mathcal{D}$-sequence ending at X is in $\mathcal{E}$. If $\mathcal{D}$ consists of all short exact sequences and $\mathcal{E}$ of the pure ones, $\mathcal{E}$-divisibility is absolute purity. They show that both halves of a relative cotorsion pair arise in this way for suitable exact structures, and they work out the cases of exact structures generated by simple modules and by modules with zero Jacobson radical.

Approximations by divisible modules are now well understood. Angeleri Hügel, Herbera and Trlifaj [7], related divisible modules to localisations and ring epimorphisms. Bazzoni and Salce [11], proved that over any commutative domain every h-divisible module has a strongly flat cover, while all divisible modules have strongly flat covers if and only if the domain is a Matlis domain. Fuchs [36], examined covers and envelopes for several classes defined by divisibility, and Zhang [95, 96], carried both theorems of Bazzoni and Salce over to commutative rings with a multiplicative subset. The paper of the second author on divisibility and co-divisibility in modules [68], continues Bland's line.

For the two torsion-theoretic classes themselves, the approximation question has a simple answer, which we record because it is easy to overlook.

Proposition 4.4: Let τ be a hereditary torsion theory.

1. For every module M, the inclusion of M into its τ-injective hull $E_\tau(M)$ is a τ-injective envelope. In particular the τ-injective modules, which form the class $\{R/I : I \in \mathfrak{F}\}^{\perp_1}$, are always enveloping.
2. The τ-divisible modules (in the sense $ID = D$ for $I \in \mathfrak{F}$) are closed under quotients, extensions and direct sums. Hence every module M has a largest τ-divisible submodule $d_\tau(M)$, and $d_\tau(M) \hookrightarrow M$ is a cover by τ-divisible modules.

Proof. (1) Recall that $E_\tau(M)$ is the preimage of $\tau(E(M)/M)$ in the injective hull $E(M)$. It is τ-injective, and M is both essential and τ-dense in it. Since maps into τ-injective modules extend along τ-dense inclusions, $M \to E_\tau(M)$ is a preenvelope. Let f be an endomorphism of $E_\tau(M)$ that is the identity on M. Then $\ker f \cap M = 0$, so $\ker f = 0$. The image of f is τ-injective and contains M, so it is τ-dense in $E_\tau(M)$. It is therefore a direct summand, and since it is also essential it equals $E_\tau(M)$. (2) Closure under quotients and direct sums is clear. For an extension $0 \to D' \to D \to D'' \to 0$ of τ-divisible modules, $ID + D' = D$ and $D' = ID' \subseteq ID$, so $ID = D$. A sum of τ-divisible submodules is a quotient of their direct sum, so $d_\tau(M)$ exists. Any map from a τ-divisible module into M has image inside $d_\tau(M)$, which gives the cover property. ▫

So the question of envelopes and covers for these classes is trivial. The real content lies one step further, in the left-hand classes of the associated cotorsion pairs and in the torsion-theoretic analogue of strongly flat modules, as in the work of Bazzoni and Salce. Problem 11.3 is posed in that form.

4.3. Copurity and Cogenerated Classes

Dual to the projectively generated proper classes are the *injectively cogenerated* ones. For a class $\mathcal{C}$ of modules these consist of the sequences $\mathbb{E}$ for which $\mathrm{Hom}_R(\mathbb{E}, C)$ is exact for all $C \in \mathcal{C}$. A warning is in order

about the word *copure*, which is used in the literature in (at least) two different senses.

The first sense is that of Enochs and Jenda [29]. A module M is *copure injective* if $\mathrm{Ext}^1_R(E,M)=0$ for every injective E, and *copure flat* if $\mathrm{Tor}^R_1(E,M)=0$ for every injective right module E. Enochs and Jenda showed that copure injective preenvelopes exist over Noetherian rings, and copure flat preenvelopes over commutative Artinian rings. They also showed that a two-sided Noetherian ring must be 2-Gorenstein if the strongly copure injective modules form a subcategory closed under cokernels, or if the strongly copure flat modules form one closed under kernels. More recently Maurya and Toksoy [53], studied copure-direct-injective modules, defined by asking that copure submodules isomorphic to direct summands be direct summands themselves. They compared them with pure-direct-injective modules (Section 7.2).

In the second sense, copurity means the purity cogenerated by a chosen class, for instance the cocyclic (subdirectly irreducible) modules. Every tensor-type purity is of this kind. For a right module X and a short exact sequence $\mathbb{E}$ of left modules, $(X\otimes_R \mathbb{E})^+ \cong \mathrm{Hom}_R(\mathbb{E}, X^+)$ naturally, and a sequence of abelian groups is exact if and only if its character dual is exact. So we have:

Remark 4.5: *For any class $\mathcal{S}$ of right modules, a sequence is $\mathcal{S}$-flatly pure if and only if it is injectively cogenerated by $\{X^+ : X\in\mathcal{S}\}$. In particular cyclic purity is the purity cogenerated by the modules $(R/I)^+$, I a right ideal.*

The cyclic–cocyclic pairing therefore comes about naturally. Over a quasi-complete Noetherian local ring, Divaani-Aazar, Esmkhani and Tousi [25], proved that the cyclically pure injective modules are exactly the direct summands of products of cocyclic modules. So cyclic purity is a purity whose relative injectives are made from cocyclic modules. This is the pairing in the title of [72], and it is also behind the pairing of T-purity with T-copurity in [67]. The version of copurity attached to a multiplicative set appears in Section 6.1.

5. Regularity, Ideal Purity and Conditions of Σ-Type

5.1. Regular Modules and Ideal Purity

Fieldhouse called a module regular if all of its submodules are pure, and proved that every module is regular exactly when R is von Neumann regular [35]. In the same paper he introduced the regular socle (the sum of the regular submodules) and used localisation to settle a conjecture of Bass on commutative perfect rings. Any other purity gives a regularity of its own. One may call a module $\mathcal{P}$-regular if every short exact sequence with middle term in a prescribed class lies in $\mathcal{P}$, and call the ring $\mathcal{P}$-regular if every module is. RD-purity, cyclic purity and S-purity give a hierarchy of weak regularity conditions in this way, and much of the work cited in this section is about such hierarchies.

Another specialisation tests ideals only. For commutative R, call $N\le M$ *ideal-pure* if $N\cap IM = IN$ for all ideals I. This is the commutative form of cyclic purity, and it is the purity used in the recent papers on P-essential and pure essential submodules [1-45]. Since principal ideals are among those tested, ideal purity implies RD-purity, and over a Prüfer domain it therefore coincides with Cohn purity by Warfield's theorem. Over other domains it is in general weaker; one sign of this is that pure-injective and cyclically pure injective modules differ over every non-Prüfer domain [26]. The absolute version (a module that is ideal-pure in every module containing it) lies between injectivity and absolute purity. The second author's paper on ideal purity and absolute purity [59], compares precisely these two absolute notions.

5.2. Local Projectivity and Its Variants

Zimmermann-Huisgen [100], called a module P *locally projective* if the following holds for every epimorphism $g: M\to L$ and every map $h: P\to L$: for each finitely generated $F\le P$ there is $h_F: P\to M$ with $g\,h_F$ equal to h on F. Projective modules are locally projective, finitely generated locally projective modules are projective, and locally projective modules are torsionless. The notion came out of her description of the pure submodules of direct products of free modules. Local projectivity is closely related to the Mittag-Leffler conditions of Raynaud and Gruson. Angeleri Hügel and Herbera [6], developed these conditions in a relative form and showed how they govern approximations by cotorsion pairs; relative Mittag-Leffler modules are discussed again in Section 8.5.

5.3. Two Settings for Σ-Type Conditions

In [66], the second author studied σ-purity along with σ-regular rings and modules. The papers [69], and [65], deal with σ-projectivity and σ-semisimplicity, and with an (FG, σ)-version of pure flatness and local σ-projectivity. All of these are relative forms of four classical notions: Fieldhouse regularity, projectivity, pure semisimplicity and local projectivity in the sense of Zimmermann-Huisgen. There are two standard settings in which such relative notions can be formulated, and since the problems of Section 11 make sense in both, we describe both.

a. *The category $\sigma[M]$.* For a module M, Wisbauer's category $\sigma[M]$ is the full subcategory of R-Mod whose objects are the submodules of M-generated modules [89]. It is a locally finitely generated Grothendieck category, and purity in $\sigma[M]$ is defined using the finitely presented objects of $\sigma[M]$, not those of R-Mod. Regularity, pure semisimplicity and local projectivity all make sense inside $\sigma[M]$. For $M=R$ they are the classical notions; for other M the theory can change, because a

finitely presented object of $\sigma[M]$ need not be a finitely presented R-module.

b. *A hereditary torsion theory.* Given a hereditary torsion theory $\sigma = (\mathcal{T}, \mathcal{F})$, one can test purity only against σ-torsion-free (or σ-closed) objects, or pass to the quotient category $R\text{-Mod}/\mathcal{T}$. This leads to σ-flatness, σ-finitely presented modules, and the relative Noetherian and coherent conditions of Stenström's theory of rings of quotients.

In both settings one expects a version of Theorem 2.3(1), saying that every object is σ-regular if and only if the ring (or the subcategory) satisfies a regularity condition; the difficulty is to find the finiteness assumptions needed for the converse. For semisimplicity the situation is clearer than it first appears. As Proposition 7.1 below shows, for a purity generated by a set of finitely presented modules, relative pure semisimplicity reduces to the absolute notion, so it is the "finitely generated" version (Corollary 7.2 and Problem 11.1) that carries new information.

6. Purity Relative to a Multiplicative Subset

The direction that has grown fastest in the last few years replaces exactness by exactness "up to an element of S", for a multiplicative subset S of a commutative ring R. The idea comes from the S-Noetherian rings of Anderson and Dumitrescu [5]: every ideal I contains a finitely generated ideal K such that $sI \subseteq K$ for some $s \in S$. At least three versions of purity have been built on this idea. They behave differently and should not be confused.

6.1. S-Pure and S-Copure Submodules

Farshadifar [32] calls a submodule $N \leq M$ *S-pure* if there is an $s \in S$ with

$$s\,(N \cap IM) \subseteq IN \qquad \text{for every ideal } I \text{ of } R,$$

And calls M *fully S-pure* if every submodule of M is S-pure. For $S = \{1\}$ this is ideal purity (Section 5.1). A good part of the basic theory survives. Under suitable hypotheses the S-pure submodules of M containing N correspond to the S-pure submodules of M/N, and when S satisfies a maximal multiple condition, chains and intersections of S-pure submodules behave well. The dual notion is in [33]. There N is *S-copure* in M if for some $s \in S$

$$s\,(N{:}_M I) \subseteq N + (0{:}_M I) \qquad \text{for every ideal } I,$$

where $(N{:}_M I) = \{m \in M : Im \subseteq N\}$. The pair of S-purity and S-copurity plays, for multiplicative sets, the part that purity and copurity, or T-purity and T-copurity, play in Section 4.

6.2. Uniform S-Versions of Absolute Purity and Flatness

A second, homological, version is due mainly to Zhang and Qi. A module T is *u-S-torsion* if $sT = 0$ for some $s \in S$. A sequence is *u-S-exact* if its homology is u-S-torsion, with one element of S serving at every position. A module F is u-S-flat if $\mathrm{Tor}_1^R(M, F)$ is u-S-torsion for all M, and R is called u-S-von Neumann regular when every module is u-S-flat [92]. In [93] Zhang introduced u-S-pure sequences. He called E *u-S-absolutely pure* if every u-S-exact sequence beginning with E is u-S-pure, and proved the following.

Theorem 6.1 [93]: *Let R be commutative and S a multiplicative subset.*

1. *E is u-S-absolutely pure if and only if some $s \in S$ satisfies $s\mathrm{Ext}_R^1(F, E) = 0$ for all finitely presented F.*
2. *R is u-S-von Neumann regular if and only if every R-module is u-S-absolutely pure.*
3. *If S consists of non-zero-divisors, then R is uniformly S-Noetherian if and only if every u-S-absolutely pure module is u-S-injective.*

Parts (1) and (2) are the uniform forms of parts (1) and (2) of Theorem 2.3. In the non-uniform version the element of S may vary with the test module. It is studied in [94], where s-pure sequences and s-absolutely pure modules are defined. There S-coherent rings are characterised by the property that pure quotients of absolutely pure modules, or direct limits of absolutely pure modules, are s-absolutely pure, which is the analogue of Theorem 2.3(3). A survey of the uniform theory by Zhang and Qi [98], also covers u-S-global and weak global dimensions.

6.3. Strong Purity and S-Phantom Morphisms

The third version, due to Hafezi, Asadollahi, Sadeghi and Zhang [39], is based on strongly flat modules. Let R_S denote the localisation. In their terminology a short exact sequence $\mathbb{E}$ is *S-pure* if it is pure, $\mathbb{E} \otimes_R R_S$ splits over R_S, and $\mathbb{E} \otimes_R R/sR$ splits over R/sR for each $s \in S$. Every sequence ending in an S-strongly flat module has these properties. The authors show that S-pure sequences give a subfunctor of Ext, that every module has an S-pure injective envelope, and that the ideal of S-phantom morphisms (the morphism version of S-strong flatness) is special precovering. The background is the work of Positselski and Slávik [76], and of Bazzoni and Positselski [10], on S-strongly flat and S-weakly cotorsion modules, together with Herzog's phantom morphisms [42]. Some results of [39], depend on the *Optimistic Conjecture* of Positselski and Slávik: if $\mathrm{pd}_R R_S \leq 1$, then a flat module F is S-strongly flat as soon as F_S is projective over R_S and F/sF is projective over R/sR for all $s \in S$. Hafezi *et al*., prove a version of it for ideals of morphisms; the module version is still open (Problem 11.4).

The three notions differ in kind. Farshadifar's S-purity weakens ideal purity; Zhang's u-S-purity weakens Cohn purity by allowing errors annihilated by an element of S; and the strong purity of Hafezi *et al*., strengthens Cohn purity by requiring splitting after

localisation and after reduction modulo s. The uniform version is the convenient one for transferring chain conditions, the strong version for approximation theory.

7. Pure-Injective and Pure-Projective Objects

7.1. Pure Semisimplicity and Approximations

A ring is *left pure-semisimple* if every left module is pure-projective; equivalently, every left module is pure-injective, or every left module is a direct sum of finitely generated modules. Zhang and Qi [97], list fourteen equivalent conditions. Among them are the existence of pure-projective (pre)envelopes, or of pure-injective (pre)covers, for all modules; closure of pure-projective modules under (countable) products or under direct limits; and closure of pure-injective modules under (countable) direct sums, direct limits or inverse limits. The proof uses the functor-category description of pure-injective modules and Šaroch's solution of Enochs' conjecture for small precovering classes [81]. The *pure semisimplicity conjecture* says that a left pure-semisimple ring has finite representation type. It holds for Artin algebras and some other classes of rings but is open in general [77].

One might hope that relative versions of pure semisimplicity, for purities generated by smaller test classes, would be easier to handle. For projectively generated purities this hope is unfounded, because of the following observation. For a set $\mathcal{G}$ of finitely presented left modules let $\mathcal{P}_{\mathcal{G}}$ be the class of short exact sequences on which $\mathrm{Hom}_R(G,-)$ is exact for every $G \in \mathcal{G}$, and call R *left $\mathcal{G}$-pure-semisimple* if every sequence in $\mathcal{P}_{\mathcal{G}}$ splits.

Proposition 7.1

R is left $\mathcal{G}$-pure-semisimple if and only if R is left pure-semisimple and every finitely presented left module lies in $\mathrm{add}(\mathcal{G} \cup \{R\})$. In that case $\mathcal{P}_{\mathcal{G}}$ is Cohn purity, and every left module is a direct sum of indecomposable modules from $\mathrm{add}(\mathcal{G} \cup \{R\})$.

Proof. The $\mathcal{P}_{\mathcal{G}}$-projective modules are the direct summands of direct sums of modules from $\mathcal{G} \cup \{R\}$ [84]. Suppose every sequence in $\mathcal{P}_{\mathcal{G}}$ splits, so that every module is $\mathcal{P}_{\mathcal{G}}$-projective, and let F be finitely presented. Then F is a summand of some $\bigoplus_i G_i$. Since F is finitely generated, its image lies in a finite subsum, and the projection onto F restricted to that subsum is still a retraction. So $F \in \mathrm{add}(\mathcal{G} \cup \{R\})$. Consequently the test classes of $\mathcal{P}_{\mathcal{G}}$ and of Cohn purity (in the form (P2)) have the same additive closure, the two proper classes coincide, and every pure sequence splits. The converse is clear. For the last statement, a left pure-semisimple ring is left Artinian and its modules are direct sums of finitely generated indecomposable modules [77]; these are finitely presented and so lie in $\mathrm{add}(\mathcal{G} \cup \{R\})$. ▫

So, for instance, RD-pure-semisimplicity or n-pure-semisimplicity tells us nothing that pure semisimplicity does not, and the pure semisimplicity conjecture cannot be approached through such relative versions. A finitely generated version, however, does carry new information.

Corollary 7.2

Every finitely generated left module is $\mathcal{P}_{\mathcal{G}}$-projective if and only if R is left Noetherian and $\mathcal{P}_{\mathcal{G}}$ coincides with Cohn purity. In particular, a commutative domain has every finitely generated module RD-projective if and only if it is a Dedekind domain.

Proof. By the argument above, a finitely generated $\mathcal{P}_{\mathcal{G}}$-projective module is a summand of a finite direct sum of finitely presented modules, hence finitely presented. So every cyclic module is finitely presented and R is left Noetherian. As every finitely presented module is $\mathcal{P}_{\mathcal{G}}$-projective, sequences in $\mathcal{P}_{\mathcal{G}}$ satisfy (P2), and $\mathcal{P}_{\mathcal{G}}$ is Cohn purity. Conversely, over a left Noetherian ring finitely generated modules are finitely presented and hence pure-projective. For the last sentence combine this with Warfield's theorem [88]: a Noetherian Prüfer domain is Dedekind. ▫

Corollary 7.2 turns relative semisimplicity for finitely generated modules into a question about coincidence of purities over Noetherian rings. For cyclic purity over commutative Noetherian rings, questions of this kind have a long history in commutative algebra; the one relevant here is Problem 11.1.

Pure-projective modules can have a complicated structure even when the ring is far from pure-semisimple. Herbera, Příhoda and Wiegand [40], looked at countably generated pure-projective modules over one-dimensional local domains. They found that $\mathrm{Add}(M)$ can contain many modules that are not direct sums of finitely generated ones, and they compared the monoid of such modules with the corresponding monoid over the completion by means of trace ideals of endomorphism rings. Turning to ring constructions, Mao [51], described pure-projective, pure-injective and FP-injective modules over a trivial extension $R \ltimes M$ and applied this to Morita context rings. Zeng [91] considered the modules of finite pure-injective (or pure-projective) dimension. He proved that inside these classes a small submodule can always be enlarged to a pure submodule of bounded size that still lies in the class, and deduced that every module has a preenvelope by modules of bounded pure-injective dimension. Results of this kind are governed by a general principle of Holm and Jørgensen [43]: a class closed under pure quotients and direct sums is covering, and a class closed under pure submodules and products is preenveloping. Göbel and Trlifaj [37], give a full account of these methods.

7.2. Direct-Injective and Direct-Projective Variants

A module is C2 (or direct-injective) if each submodule isomorphic to a direct summand is itself a direct summand. If only pure submodules are considered

one gets *pure-direct-injective* (pure-C2) modules [52]. Maurya, Das and Alagöz proved that a right Noetherian ring over which every pure-C2 right module is pure-injective must be Krull–Schmidt semiperfect. They also found that every pure-C2 module is injective (or projective, or dual-Utumi) only over semisimple rings, and that R is right perfect exactly when all projective right modules are pure-C2. On the dual side, Alizade and Toksoy [4], defined *pure-direct-projective* modules: whenever A is a pure submodule and M/A is isomorphic to a summand of M, A must be a summand. Injective modules over a Noetherian ring are pure-direct-projective, and a right hereditary ring is right Noetherian if and only if every injective right module is pure-direct-projective. The copure version of [53], was described in Section 4.3. Taken together, these three papers suggest a common framework: for an arbitrary proper class $\mathcal{P}$ one can define $\mathcal{P}$-direct-injective and $\mathcal{P}$-direct-projective modules and ask for which rings they coincide with classical classes (Problem 11.7).

8. Purity beyond Module Categories

8.1. Locally Finitely Presented Categories and the Ziegler Spectrum

Purity can be defined in any additive category with direct limits that is generated by its finitely presented objects. Crawley-Boevey [22], did this for locally finitely presented additive categories $\mathcal{A}$: a sequence is pure exact if it stays exact under $\mathrm{Hom}(C, -)$ for every finitely presented C. Pure-injective objects then correspond to injective objects of a functor category. The *definable subcategories*, i.e. those closed under products, direct limits and pure subobjects, correspond to the closed subsets of the Ziegler spectrum. Ziegler [99], had defined this spectrum, in model theory, as the set of isomorphism classes of indecomposable pure-injective modules with a suitable topology. Herzog [41], constructed it for locally coherent Grothendieck categories, and Prest's book [77], is the standard reference.

8.2. Exact Structures and Relative Purity

Schlegel [82], connected purity in a locally finitely presented category $\mathcal{A}$ with products to exact structures on the full subcategory $\mathrm{fp}\mathcal{A}$ of finitely presented objects. He obtained a bijection between exact structures on $\mathrm{fp}\mathcal{A}$ and the definable subcategories that contain all objects which are fp-injective relative to the largest exact structure; equivalently, closed subsets of the Ziegler spectrum containing all indecomposables of that kind. For an Artin algebra Λ the result gives a bijection between indecomposable endofinite non-injective modules and the maximal exact structures on $\mathrm{mod}\Lambda$ that admit an almost exact sequence. Under this bijection the generic modules correspond to the exact structures containing all almost split sequences. Positselski [75], proved that an exact structure on a small idempotent-complete additive category extends in exactly one way to a locally coherent exact structure on its category of ind-objects. Hence every relative purity of the kind considered in Section 3 is the restriction of a well-behaved exact structure on a larger category. Krause [48], defined a Boolean spectrum for an arbitrary Grothendieck category. With it he extended Crawley-Boevey's correspondence from definable subcategories to subcategories closed under coproducts, pure subobjects and pure-injective envelopes, and he showed that for an Artin algebra the extension brings nothing new exactly when the algebra has finite representation type.

8.3. Geometric and Monoidal Purity

On a scheme, and more generally in a closed symmetric monoidal Grothendieck category, there are two notions of purity: a categorical one using finitely presented objects and a geometric one using the tensor product. Estrada, Gillespie and Odabaşı [31], studied pure exact structures on categories of quasi-coherent sheaves and the pure derived categories they give rise to. Holm and Odabaşı [44], showed that for a Grothendieck cosmos the geometrically pure exact category has enough relative injectives, and in fact geometrically pure injective envelopes. In many cases of interest they identified the geometrically pure injective objects with the injective objects of a category of enriched functors.

8.4. Triangulated Categories

A compactly generated triangulated category T has no non-split monomorphisms, so purity has to be defined differently. One uses the restricted Yoneda functor into modules over the compact objects, and calls a triangle pure if its image is a short exact sequence. Bird and Williamson [14], developed duality pairs and phantom maps in this setting and applied them to definable subcategories. In [15], they proved that the homological and Balmer spectra of tensor-triangular geometry are functorial for definable functors, which are exactly the functors that preserve purity. Since purity in T depends only on the additive structure, many module-theoretic arguments carry over almost unchanged.

8.5. Mittag-Leffler Conditions

A right module M is *Mittag-Leffler* if $M \otimes_R \prod_i Q_i \to \prod_i (M \otimes_R Q_i)$ is injective for every family (Q_i) of left modules, and $\mathcal{Q}$-*Mittag-Leffler* if this is asked only for families taken from a class $\mathcal{Q}$. The countably generated Mittag-Leffler modules are exactly the countably generated pure-projective modules. Rothmaler [80], showed that this description does not survive in the relative setting. Instead, he described the countably generated relatively Mittag-Leffler modules as direct summands of preenvelopes of unions of relatively pure chains of finitely presented modules, and he used RD-purity as a test case. Ben Yassine and Trlifaj [13], characterised flat relative Mittag-Leffler modules by their local structure, and proved Enochs' conjecture for all of these classes. Prest [78], studied strictly atomic modules in definable categories, which is the model-theoretic side of the same phenomenon.

9. Primeness in Modules and in Near-Rings

9.1. Prime Submodules Relative to a Multiplicative Set

Primeness and purity meet in several recent papers, partly because both use multiplicative sets and partly through classes of modules whose pure submodules inherit some property. Let R be commutative and $N \leq M$ with $(N:_R M) \cap S = \emptyset$. Sevim, Arabacı, Tekir and Koç [83], call N *S-prime* if there is a fixed $s \in S$ such that $am \in N$ (with $a \in R$, $m \in M$) always forces $sa \in (N:_R M)$ or $sm \in N$. In the *weakly S-prime* submodules of Khashan and Yetkin Çelikel [47], this is only required when $am \neq 0$. Yetkin Çelikel and Khashan [90], did the same for primary submodules. Both papers work out how the notions behave under localisation, quotients, direct products and idealisation. Baupradist and Hukaew [9], studied S-prime submodules of S-multiplication modules. Among other things they observed that a pure submodule of an S-multiplication module is again an S-multiplication module, a small example of purity carrying structure from a module to its submodules.

9.2. Prime Ideals of Near-Rings

A (right) near-ring is a set N with an addition making it a group (not necessarily abelian) and a multiplication making it a semigroup, linked by only one distributive law [74]. Without the second distributive law the notion of a prime ideal splits into several inequivalent ones. An ideal P is *0-prime* if $IJ \subseteq P$ for ideals I, J implies that $I \subseteq P$ or $J \subseteq P$. It is *3-prime* if $aNb \subseteq P$ implies that $a \in P$ or $b \in P$, *completely prime* if $ab \in P$ implies the same, and *equiprime* if for $a \in N \backslash P$ and $x, y \in N$

$$anx - any \in P \text{ for all } n \in N \quad \Rightarrow \quad x - y \in P.$$

Equiprime ideals were introduced by Booth, Groenewald and Veldsman [17], in order to obtain a prime radical with the Kurosh–Amitsur properties. It was a long-standing question whether every zero-symmetric simple near-ring with identity is equiprime. Ke and Meyer [46], answered it in the negative, with counterexamples built from HNN extensions of infinite simple groups.

The list of prime notions keeps being refined. Taşdemir, Atagün and Altındiş [87] studied prime-type N-ideals of N-groups (the near-ring version of modules) and defined IFP N-ideals. Muhammad, Ashraf and Khan [58], defined v-primary ideals for $v \in \{0,1,2,3,c\}$ and attached graphs to them. Chaithra, Kedukodi, Aishwarya, Koppula and Kuncham [19], introduced prime product fractal ideals, and Mohammadi and Davvaz [56], developed prime, primary and subtractive ideals for (m,n)-near rings. Djagba and Juglal [27], transferred Behboodi's classical primeness to near-ring modules; unlike in the ring case, they had to separate several inequivalent versions. The second author's papers on prime ideals and ideal symmetry, and on radicals of ideals, in near-rings [62, 63], and on $-submodules and prime $-submodules of an M-module [60, 61], belong to the same effort to distinguish the prime notions and to follow them from the ring to its modules.

As far as we know, nobody has yet developed a theory of purity for near-ring modules. Tensor products are not available, but conditions (P2) and (P3) of Section 2 still make sense; this is the starting point of Problem 11.9.

10. A Quantitative Look at the References

The hundred references were chosen because they bear on the questions discussed above, not by a systematic search of a database. The numbers below therefore describe this collection and not the field as a whole, but they do show where the activity is. Table 2 sorts the references by the eight themes of Sections 2 to 9 and by period.

Table 2: Distribution of the 100 references by theme and period of publication

Theme	up to 1999	2000–2021	2022–2026	Total
A. Foundations and absolute purity (Section 2)	8	1	1	10
B. Purities generated by test modules (Section 3)	1	9	7	17
C. Torsion-theoretic purity, divisibility, copurity (Section 4)	2	8	5	15
D. Regularity and σ-type conditions (Section 5)	3	4	0	7
E. Purity relative to multiplicative subsets (Section 6)	0	4	7	11
F. Pure-injective and pure-projective objects (Section 7)	0	2	7	9
G. Purity in categories, Mittag-Leffler conditions (Section 8)	3	4	9	16
H. Primeness in modules and near-rings (Section 9)	2	6	7	15
Total	**19**	**38**	**43**	**100**

Of the hundred items, forty-three are from 2022 or later: thirty-three journal articles and ten preprints, the preprints mostly in themes E and G, where things move fastest. By year, six of the recent items are from 2022, ten from 2023, eighteen from 2024, six from 2025 and three from 2026. The drop after 2024 is probably an artefact, as recent work takes time to appear in print and in the indexes.

The two themes that grew fastest sit at opposite ends of the subject. One is the "local" theory attached to a multiplicative subset (theme E, seven of eleven items

recent); the other is the "global" theory of purity in categories (theme G, nine of sixteen). They hardly cite each other. The uniform S-theory is written in terms of commutative algebra and vanishing of Ext, the categorical theory in terms of functor categories and exact structures, and Problem 11.5 asks for a link between them.

Theme D, on the other hand, has nothing from the last four years. The σ-type papers of the second author [65-69], are among its most recent entries. Proposition 7.1 shows that part of this ground is less open than it looks, since relative pure semisimplicity for projectively generated purities reduces to the absolute case. The finitely generated and torsion-theoretic versions (Problems 11.1 and 11.2) remain.

Theme B (purities generated by cyclic, simple and finitely n-presented modules) has produced new results at a fairly constant rate for twenty years. A large share of it comes from a few research groups, in particular in Türkiye, Iran and India, and is published in journals such as the *Journal of Algebra and Its Applications*, *Communications in Algebra* and the *Lobachevskii Journal of Mathematics*. The near-ring papers of theme H are concentrated in a similar way. Two of them appeared in the *Indian Journal of Pure and Applied Mathematics* [19-87], the second by an Indian group, and the others are by authors from Türkiye, Pakistan, Iran and South Africa.

11. Open Problems

Each problem below comes with a short explanation of why it seems worth studying and a possible way to begin. Some are older questions put into a relative form; others come directly out of the material above, including the observations in Propositions 3.1, 4.3, 4.4 and 7.1.

Problem 11.1 (Finitely Generated Relative Semisimplicity)

Characterise the commutative Noetherian rings over which every finitely generated module is cyclically pure-projective; by Corollary 7.2, these are the rings over which cyclic purity coincides with purity. Is every such ring arithmetical, so that RD-purity already coincides with purity?

Motivation

Proposition 7.1 shows that the relative pure semisimplicity question for projectively generated purities is the classical pure semisimplicity conjecture in disguise [77]. Corollary 7.2 shows that the finitely generated version is different: it singles out Dedekind domains in the RD case and leads, for cyclic purity, to a genuine question about Noetherian rings. The σ-semisimplicity of [69], and the list of characterisations of pure-semisimple rings in [97], suggest checking which of those characterisations have "finitely generated" analogues.

Approach

Both purities are local, so one may assume R local and then pass to its completion. Cyclic purity tests only the modules R/I, and the simplest finitely presented modules it does not test directly are those of the form $R^2/R(a,b)$. So the first step is to decide, over a two-dimensional regular local ring, whether a cyclically pure sequence must stay exact after tensoring with such modules. A counterexample there would suggest that the answer is "arithmetical rings only". Matlis duality translates the question into one about Artinian modules, which connects it with Problem 11.8.

Problem 11.2 (τ-Prüfer Rings)

Call R *τ-Prüfer* if every τ-torsion-free module is flat. Characterise the τ-Prüfer rings for the Goldie, Lambek and Dickson torsion theories. For a commutative ring R and a torsion theory τ of finite type with $\tau(R) = 0$, is R τ-Prüfer if and only if the ring of quotients R_τ is von Neumann regular and every finitely generated ideal I with $R_\tau I = R_\tau$ is projective?

Motivation

By Proposition 4.3, these are exactly the rings over which τ-purity and Cohn purity agree on torsion-free modules (after passing to $R/\tau(R)$), so the question is the general form of Warfield's theorem on Prüfer domains [88]. The torsion purity and M-purity of [70, 71], are special cases.

Approach

Over a left Noetherian ring every module is the directed union of its finitely generated submodules, and submodules of torsion-free modules are torsion-free; so R is τ-Prüfer exactly when finitely generated τ-torsion-free modules are projective. That reduces the Noetherian case to finitely generated modules. In general, start with $\tau = \tau_S$ for a set S of non-zero-divisors, where $R_\tau = R_S$, and compare the conjectured conditions with what is known for Prüfer domains. For the Goldie theory, the question becomes when finitely presented nonsingular modules are projective.

Problem 11.3 (τ-Strongly Flat Covers)

Let R be commutative and τ of finite type with $\tau(R) = 0$ and $\mathrm{pd}_R R_\tau \leq 1$. Call a module *$\tau$-strongly flat* if it lies in ${}^{\perp_1}\left(R_\tau^{\perp_1}\right)$. Do the τ-strongly flat modules form a covering class if and only if R_τ is perfect and R/I is perfect for every $I \in \mathfrak{F}$? What are the corresponding statements for the co-divisible modules of [16-68]?

Motivation

Proposition 4.4 shows that the τ-injective and the τ-divisible classes always admit envelopes or covers, so for them the approximation question is empty. The meaningful question concerns the left-hand class of the cotorsion pair cogenerated by R_τ. For the classical

torsion theory of a domain this is the theorem of Bazzoni and Salce on strongly flat covers and almost perfect domains [11]. For $\tau = \tau_S$ it is the theorem of Bazzoni and Positselski [10], and Zhang's work on S-divisible modules [96], treats related cases. A common formulation for Gabriel topologies would cover all of these.

Approach

The "only if" direction should follow the arguments of [10, 11], which use Bass's characterisation of perfect rings rather than Enochs' conjecture. For the "if" direction one wants to show that over a ring satisfying the perfectness conditions every flat module is τ-strongly flat, using the decomposition into the part over R_τ and the parts over the rings R/I. Here the tilting classes coming from localisations [7], and Šaroch's results on Enochs' conjecture [81] should help. The cotorsion pair cogenerated by $\{R/I : I \in \mathfrak{F}\}$ is complete by the Eklof–Trlifaj theorem [37], and that is the natural place to start.

Problem 11.4 (The Optimistic Conjecture and Strong Purity)

Let R be commutative and S a multiplicative subset with $\mathrm{pd}_R R_S \leq 1$. If F is flat, F_S is projective over R_S, and F/sF is projective over R/sR for every $s \in S$, must F be S-strongly flat?

Motivation

This is the Optimistic Conjecture of Positselski and Slávik [76]. Several results on strong purity and S-phantom morphisms depend on it [39], among them the statement that S-pure injective modules are closed under extensions if and only if every S-weakly cotorsion module is S-pure injective.

Approach

The conjecture is known when S is countable, when S consists of non-zero-divisors, when S is bounded, and when R is S-h-nil [10=76]; Hafezi *et al*., also have a version for morphisms. The class of flat modules satisfying the two local conditions is closed under transfinite extensions, since flatness makes both localisation and reduction modulo s exact along a filtration. A Hill-lemma argument should then reduce the conjecture to κ-presented modules for $\kappa = |R| + \aleph_0$. The hard step would be to go down further to countably presented modules, where the countable case applies.

Problem 11.5 (A Spectrum for Uniform S-Purity)

Develop a theory of u-S-pure-injective modules. Do u-S-pure-injective envelopes exist? Is there a topological space whose closed subsets correspond to the classes closed under products, direct limits and u-S-pure submodules?

Motivation

The uniform S-theory reproduces the module-theoretic part of Theorem 2.3 [93-98]. It does not yet have a model-theoretic part, namely the Ziegler spectrum and definable subcategories [22-99], and without that part there is no uniform analogue of the decomposition theory of pure-injective modules.

Approach

A workable definition is to call E u-S-pure-injective if a single $s \in S$ annihilates $\mathrm{Pext}^1_R(C, E)$ for all C. One checks directly that this happens if and only if multiplication by s on E factors through the pure-injective hull of E. That gives a concrete handle on envelopes. For the spectrum, note that the u-S-torsion modules form a Serre subcategory, but not a localising one, since they are not closed under infinite direct sums. One could instead work with the exact structure on $\mathrm{mod}R$ generated by the sequences that split after multiplication by an element of S. Schlegel's correspondence [82], then gives a closed subset of the Ziegler spectrum, and the question is whether it is the right space.

Problem 11.6 (Products of Relatively Flat Modules for Non-Finitely Presented Test Modules)

Let $\mathcal{S}$ be a class of right modules that are not necessarily finitely presented, for instance all simple right modules. Characterise the rings over which the left modules C with $\mathrm{Tor}_1^R(X, C) = 0$ for all $X \in \mathcal{S}$ are closed under direct products.

Motivation

For a set of finitely presented test modules Proposition 3.1 gives a complete answer: the test modules must be FP_2. Many of the purities in Section 3.3, however, are defined by simple modules, and simple modules are finitely presented only when the corresponding maximal one-sided ideals are finitely generated. The max-flat and neat-flat modules of [2-18], are of this kind, and for them the product question is open.

Approach

For a module X with a projective resolution $P_1 \to P_0 \to X \to 0$, the behaviour of $\mathrm{Tor}_1(X, -)$ on products is controlled by the canonical maps $P_i \otimes \prod Q_j \to \prod(P_i \otimes Q_j)$, that is, by Mittag-Leffler conditions on the syzygies of X relative to the class of modules in question. The relative Mittag-Leffler theory of [6-80], is therefore the natural tool, and the case of countably generated maximal ideals should be tractable first. The results should also be compared with Couchot's RD-flat modules [21] and with the (FG, σ)-pure flatness of [65].

Problem 11.7 (Direct-Injectivity Relative to a Proper Class)

For a proper class $\mathcal{P}$, call M *$\mathcal{P}$-direct-injective* if every $\mathcal{P}$-submodule of M that is isomorphic to a direct summand of M is a direct summand, and define *$\mathcal{P}$-direct-*

projective dually. For which pairs $(R, \mathcal{P})$ is every $\mathcal{P}$-direct-injective module $\mathcal{P}$-injective? When do the two notions reduce to the classical C2 and D2 conditions?

Motivation

Pure-direct-injective [52], pure-direct-projective [4], and copure-direct-injective [53], modules were studied separately, with similar methods, and in each case semisimple, perfect and Noetherian rings turn up in the answers. Even for Cohn purity it is not clear whether rings over which every pure-direct-injective module is pure-injective must be pure-semisimple. Over a left Noetherian ring every finitely generated module is pure-direct-injective, so any module witnessing the difference has to be infinitely generated.

Approach

It is easy to see that $\mathcal{P}$-quasi-injective modules are $\mathcal{P}$-direct-injective, and that a larger proper class gives a stronger condition. For the main questions, work with endomorphism rings. When $\mathcal{P}$ has injective envelopes, as Cohn purity and the strong S-purity of Section 6.3 do, $\mathcal{P}$-direct-injectivity of M can perhaps be read off from the action of $\mathrm{End}(M)$ on the envelope, in the way that automorphism-invariant modules are treated relative to injective hulls.

Problem 11.8 (Cyclic Purity and Cocyclic Copurity)
Over which rings does the proper class injectively cogenerated by the cocyclic modules coincide with cyclic purity?

Motivation

By Remark 4.5, cyclic purity is always cogenerated by the character modules $(R/I)^+$. Over a quasi-complete Noetherian local ring, Divaani-Aazar, Esmkhani and Tousi proved that the cyclically pure injective modules are summands of products of cocyclic modules [25]. The pairing of cyclic purity with co-cyclic copurity in [72], points to a duality that is not yet understood for arbitrary rings.

Approach

For a commutative Noetherian ring the cocyclic modules are the submodules of the injective hulls $E(R/\mathfrak{m})$, and both proper classes are local. Over a complete local ring, Matlis duality identifies the submodules of $E(R/\mathfrak{m})$ with the modules $\mathrm{Hom}\big(R/I, E(R/\mathfrak{m})\big)$, and from this one gets equality. Over a local ring that is not complete, the same computation shows that a sequence is cogenerated by the cocyclic modules exactly when its completion is cyclically pure over $\hat{R}$. So the problem becomes whether cyclic purity passes from R to $\hat{R}$. Outside the Noetherian case there may be too few cocyclic modules, and a counterexample may well exist (valuation domains that are not maximal are the first place to look).

Problem 11.9 (Purity for Near-Ring Modules)

Develop a theory of purity for modules over a zero-symmetric near-ring N (that is, for N-groups). In particular, find the near-rings over which every N-subgroup of every N-group is pure, as in Fieldhouse's theorem, and relate pure N-ideals to the prime notions of Section 9.2.

Motivation

Near-ring ideal theory has many prime notions [17-87], and primeness at the level of modules has been studied as well [27-61]. There is, however, no theory of purity. Tensor products are not available, so Cohn's definition cannot be used directly.

Approach

N-groups form a variety of universal algebras. In any variety a subalgebra can be called pure when it reflects the solvability of positive primitive formulas, which is the analogue of (P3). With this definition the model-theoretic methods of [77-99], suggest how to construct pure-injective hulls. The first concrete steps are to describe the finitely presented N-groups and to see whether the analogues of (P2) and (P3) agree for N-ideals. Applying purity to the N-subgroup Nn of N should show, exactly as in Fieldhouse's argument, that $n \in nNn$ whenever all N-subgroups are pure.

Problem 11.10 (Nilpotence of the τ-Pure Prime Radical)

Find conditions weaker than τ-Krull dimension under which the τ-pure prime radical of a module M is τ-nilpotent, or show that Krull dimension cannot be dropped.

Motivation

Arellano, Castro and Ríos [8], proved nilpotence for a finitely generated progenerator of $\sigma[M]$ that has τ-Krull dimension. In ring theory, Levitzki's theorem gets nilpotence of the prime radical from the ascending chain condition alone, and Goldie-type conditions on annihilators are known to suffice for nil one-sided ideals. One would expect a relative chain condition to be enough here too.

Approach

Since M is a finitely generated progenerator of $\sigma[M]$, the problem can be transferred to the endomorphism ring of M and a torsion theory on its module category. Replace Krull dimension by the ascending chain condition on τ-pure annihilator submodules and try to adapt Levitzki-type arguments to the product of submodules used in [8]. For a counterexample, look for a module with finite τ-uniform dimension but without τ-Krull dimension, for the Dickson torsion theory over a semiartinian ring that is not Noetherian.

Problem 11.11 (Relative Purity in Triangulated Categories)

Let T be a compactly generated triangulated category and $\mathcal{G}$ a set of compact objects. Call a triangle $\mathcal{G}$-pure if the restricted Yoneda functor to modules over $\mathcal{G}$ sends it to a short exact sequence. Describe the $\mathcal{G}$-pure-injective objects, the $\mathcal{G}$-definable subcategories, and the definable functors that preserve $\mathcal{G}$-purity.

Motivation

In module categories, passing from Cohn purity to purities generated by cyclic or cyclically presented modules gave the material of Sections 3 and 4. The triangulated theory [14, 15], has so far dealt only with the absolute case, where $\mathcal{G}$ is the set of all compact objects. In the derived category of a commutative ring, the Koszul objects $\mathrm{cone}(r\colon R \to R)$ give a natural analogue of RD-purity. In fact a short computation with $\mathrm{Hom}(\mathrm{cone}(r), -)$ shows that a short exact sequence of modules is pure relative to these objects exactly when it is RD-pure.

Approach

One should try to transfer Krause's construction of the purity category, together with Schlegel's bijection between exact structures and closed subsets of the Ziegler spectrum [48-82], to the triangulated setting, with exact structures replaced by Serre subcategories of finitely presented modules over the compact objects. The derived category of a Dedekind domain, where both the Ziegler spectrum and RD-purity are well understood, is a good first test.

Problem 11.12 (Deciding Purity Algorithmically)

Let R be a polynomial ring over a computable field, M a finitely generated R-module and $N \leq M$ a finitely generated submodule. Is there an algorithm that decides whether N is cyclically pure, or RD-pure, in M?

Motivation

Over a Noetherian ring a pure submodule N of a finitely generated module M is a direct summand, since M/N is finitely presented and (P2) makes the projection split. Splitting can be decided by asking whether the identity of M/N lies in the image of $\mathrm{Hom}(M/N, M)$, a membership problem that Gröbner bases solve. Cyclic purity and RD-purity, by contrast, involve infinitely many ideals or elements, and we do not know of any reduction to finitely many tests. An algorithm would also make the notions of this survey available for experiments and for teaching.

Approach

Both properties are local. Over a local ring, the Krull intersection theorem shows that it is enough to test ideals primary to the maximal ideal. The next step would be to show that a finite set of such ideals suffices, determined by the associated primes of M/N and of M, with the Artin–Rees lemma bounding the relevant powers. For $\mathbb{Z}$-modules and modules over $k[x]$ the three purities coincide and the question is settled by the Smith normal form, so the first real test is $k[x, y]$. Even there the answer may depend on the ground field, since the ideals containing a fixed power of the maximal ideal form an algebraic family.

12. CONCLUDING REMARKS

Purity first appeared as a technical tool in Cohn's work on free products of rings. Today it is used in relative homological algebra, in the model theory of modules and in the study of large objects in triangulated categories. In practice we found it most useful to treat each purity as a proper class (or exact structure) with its own injective, projective and flat objects. Results can then be carried from one purity to another, and when they fail to carry over, the failure usually reflects some arithmetic property of the ring. Propositions 3.1, 4.3, 4.4 and 7.1 illustrate this in a small way. Each was obtained by applying a classical argument (Chase's, Warfield's, the construction of the τ-injective hull, the description of pure-projective modules) to a relative setting, and in each case this either answered a question or turned it into a sharper one. The themes of the second author's work, namely torsion purity, cyclic purity, divisibility and σ-type conditions, fit naturally into this framework, and several of the problems in Section 11 ask how far they can be taken with the recent methods for multiplicative subsets, exact structures and approximation theory.

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Use of AI-Assisted Tools

In preparing this manuscript the authors used an AI assistant to help collect and check bibliographic details, to draft parts of the text and to revise its language. The authors reviewed and edited all content, checked the mathematical statements, and take full responsibility for the article.

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