Scholars Journal of Engineering and Technology (SJET) **ISSN 2321-435X (Online)**
Sch. J. Eng. Tech., 2014; 2(4A):506-510 **ISSN 2347-9523 (Print)**

(An International Publisher for Academic and Scientific Resources)
www.saspublisher.com

Research Article

Non-Homogeneous Sextic Equation with Four Unknowns $(X+Y)(X^3+Y^3)=Z^2W^4$

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Abstract: We obtain infinitely many non-zero integer quadruples(x,y,z,w) satisfying the non-homogeneous sextic equation with four unknowns. Various interesting properties among the values of x,y,z and w are presented
Keywords: Sextic equation with four unknowns, integer solutions.
2010 Mathematics Subject Classification: 11D09

INTRODUCTION

The theory of diophantine equations offers a rich variety of fascinating problems [1-4]. In [5-11] sextic equations with four unknowns are analysed for their non-zero integer solutions. This communication analyses yet another sextic equation with four unknowns given by $(x+y)(x^3+y^3)=z^2w^4$. Infinitely many non- zero integer quadruples (x,y,z,w) satisfying the above equation are obtained. Various interesting properties among the values of x,y,z and w are presented

Notations:

P_n^m - Pyramidal number of rank n with size m.

$T_{m,n}$ - Polygonal number of rank n with size m.

$F_{4,n,3}$ - Four dimensional figurative number of rank n whose generating polygonal is a triangle.

$F_{4,n,4}$ - Four dimensional figurative number of rank n whose generating polygonal is a square.

METHOD OF ANALYSIS:

The sextic equation with four unknowns to be solved is

$$(x+y)(x^3+y^3)=z^2w^4 \quad (1)$$

Introducing the linear transformations

$$x=u+v,\ y=u-v,\ z=2u \quad (2)$$

in (1), it is written as

$$u^2+3v^2=w^4 \quad (3)$$

(3) is solved through different methods and thus, we obtain different patterns of solutions to (1).

PATTERN: 1

Assume $w=(a^2+3b^2)$ (4)

Substituting (4) in (3) and employing the method of factorization, define

$$(u+i\sqrt{3}v)=(a+i\sqrt{3}b)^4$$

Equating the real and imaginary parts, we get

$$u=a^4+9b^4-18a^2b^2$$
$$v=4a^3b-12ab^3$$

In view of (2), the values of x,y, and z are given by

$$\left.\begin{aligned} x(a,b)&=a^4+9b^4-18a^2b^2+4a^3b-12ab^3\\ y(a,b)&=a^4+9b^4-18a^2b^2-4a^3b+12ab^3\\ z(a,b)&=2a^4+18b^4-36a^2b^2 \end{aligned}\right\} \quad (5)$$

Thus (4) and (5) represent the integral solutions of (1).

PROPERTIES:

- $x(1,b)-9T_{4,b^2}+24P_b^5+T_{4,b}\equiv 1(\mathrm{mod}\,1)$
- $x(a,1)-T_{4,a^2}-8P_a^5+T_{46,a}\equiv 9(\mathrm{mod}\,33)$
- $y(1,b)-9T_{4,b^2}-24P_b^5+T_{62,b}\equiv 1(\mathrm{mod}\,33)$
- $x(1,b)-216F_{4,b,3}-72P_b^5-T_{70,b}\equiv 1(\mathrm{mod}\,85)$
- $x(1,b)-108F_{4,b,4}-28P_b^5-T_{38,b}\equiv 1(\mathrm{mod}\,53)$
- $z(1,b)-432F_{4,b,3}-204P_b^5-T_{100,b}\equiv 2(\mathrm{mod}\,150)$
- $z(a,1)-48F_{4,a,3}-12P_a^5+T_{64,a}\equiv 18(\mathrm{mod}\,24)$

PATTERN: 2

Consider (3) as

$$u^2+3v^2=w^4*1 \quad (6)$$

Write 1 as

$$1=\frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4} \quad (7)$$

Substituting (7) in (6) and employing the method of factorization, define

$$(u+i\sqrt{3}v)=(a+i\sqrt{3}b)^4\,\frac{(1+i\sqrt{3})}{2}$$

Equating the real and imaginary parts, we get

$$u=\frac{1}{2}[a^4+9b^4-18a^2b^2-12a^3b+36ab^3]$$
$$v=\frac{1}{2}[a^4+9b^4-18a^2b^2+4a^3b-12ab^3]$$

In view of (2), the values of x,y, and z are given by

$$\left.\begin{aligned} x(a,b)&=a^4+9b^4-18a^2b^2-4a^3b-24ab^3\\ y(a,b)&=12ab^3-8a^3b\\ z(a,b)&=2a^4+18b^4-36a^2b^2-8a^3b-48ab^3 \end{aligned}\right\} \quad (8)$$

Thus (4) and (8) represent the integral solutions of (1).

PROPERTIES:

- $x(1,b)-9T_{4,b^2}+48P_b^5-T_{14,b}\equiv 1(\mathrm{mod}\,1)$

- $x(a,1)-T_{4,a^2}+8P_a^5+T_{30,a}\equiv 9(\text{mod }37)$
- $y(1,b)-24P_b^5+T_{24,b}\equiv 0(\text{mod }18)$
- $y(a,1)+16P_a^5-T_{18,a}\equiv 0(\text{mod }19)$
- $z(1,b)+96P_b^5-T_{26,b}\equiv 2(\text{mod }3)$
- $z(a,1)+16P_a^5-2T_{4,a^2}+T_{58,a}\equiv 18(\text{mod }21)$
- $z(a,1)-x(a,1)+8P_a^5-T_{4,a^2}+T_{30,a}\equiv 9(\text{mod }37)$

PATTERN: 3

Instead of (7), 1 can be written as

$$1=\frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{49} \qquad (9)$$

Substituting (4) and (9) in (6) and employing the method of factorization, define

$$(u+i\sqrt{3}v)=(a+i\sqrt{3}b)^4\,\frac{(1+i4\sqrt{3})}{7}$$

Equating the real and imaginary parts, we get

$$u=\frac{1}{7}[a^4+9b^4-18a^2b^2-48a^3b+144ab^3]$$
$$v=\frac{1}{7}[4a^4+36b^4-72a^2b^2+4a^3b-12ab^3]$$

In view of (2), the values of x,y, and z are given by

$$\left.\begin{aligned}x(a,b)&=\frac{1}{7}[5a^4+45b^4-90a^2b^2-44a^3b+132ab^3]\\ y(a,b)&=\frac{1}{7}[-3a^4-27b^4-54a^2b^2-52a^3b+156ab^3]\\ z(a,b)&=\frac{1}{7}[2a^4+18b^4-36a^2b^2-96a^3b+288ab^3]\end{aligned}\right\} \qquad (10)$$

Replacing a by 7A, b by 7B in the above equations (10) and (4) the corresponding integer solutions of (1) are

$$X(A,B)=1715A^4+15435B^4-30870A^2B^2-16660A^3B+45276AB^3$$
$$Y(A,B)=-1029A^4-9261B^4+18522A^2B^2-16268A^3B+53508AB^3$$
$$Z(A,B)=686A^4+6174B^4-12348A^2B^2-32928A^3B+98784AB^3$$
$$W(A,B)=49A^2+147B^2$$

PROPERTIES:

- $7X(1,B)-45T_{4,B^2}-264P_B^5\equiv 5(\text{mod }265)$
- $7X(A,1)-5T_{4,A^2}+88P_A^5+T_{94,A}\equiv 45(\text{mod }91)$
- $7Z(1,B)-18T_{4,B^2}-576P_B^5+T_{650,B}\equiv 2(\text{mod }418)$
- $7Y(1,B)+27T_{4,B^2}-312P_B^5+T_{422,B}\equiv -3(\text{mod }261)$
- $7Y(A,1)+3T_{4,A^2}+104P_A^5+T_{5,A}\equiv -27(\text{mod }156)$

PATTERN: 4

Rewrite (3) as

$$w^2 - u^2 = 3v^2 \quad (11)$$

(11) is written in the form of ratio as

$$\frac{w^2 + u}{v} = \frac{3v}{w^2 - u} = \frac{\alpha}{\beta}, \beta \neq 0$$

which is equivalent to the system of equations

$$\beta w^2 + \beta u - \alpha v = 0$$
$$\alpha w^2 - \alpha v - 3\beta v = 0$$

Applying the method of cross-multiplication, we have

$$\left.\begin{array}{l} u = \alpha^2 - 3\beta^2 \\ v = 2\alpha\beta \\ w^2 = 3\beta^2 + \alpha^2 \end{array}\right\} \quad (12)$$

Substituting $\beta = 2pq,\ \alpha = 3p^2 - q^2$ in (12) and (2), the corresponding integer solutions of (1) are

$$x(p,q) = 9p^4 + q^4 - 18p^2q^2 + 12p^3q - 4pq^3$$
$$y(p,q) = 9p^4 + q^4 - 18p^2q^2 - 12p^3q + 4pq^3$$
$$z(p,q) = 18p^4 + 2q^4 - 36p^2q^2$$

PROPERTIES:

- $x(1,q) - 24F_{4,q,3} + 20P_q^5 + T_{40,q} \equiv 9 \pmod{12}$
- $x(p,1) - 216F_{4,p,3} - 120P_p^5 - T_{22,p} \equiv 1 \pmod{53}$
- $x(1,q) - 12F_{4,q,4} + 18P_q^5 - T_{36,q} \equiv 9 \pmod{14}$
- $x(p,1) - 108F_{4,p,4} - 56P_p^5 - T_{38,p} \equiv 1 \pmod{45}$
- $z(1,q) - 48F_{4,q,3} - 12P_q^5 + T_{64,q} \equiv 18 \pmod{24}$

PATTERN: 5

Also (11) is written in the form of ratio as

$$\frac{w^2 + u}{3v} = \frac{v}{w^2 - u} = \frac{\alpha}{\beta}, \beta \neq 0$$

which is equivalent to the system of equations

$$\beta w^2 + \beta u - 3\alpha v = 0$$
$$\alpha w^2 - \alpha v - \beta v = 0$$

Applying the method of cross-multiplication, we have

$$\left.\begin{array}{l} u = 3\alpha^2 - \beta^2 \\ v = 2\alpha\beta \\ w^2 = 3\alpha^2 + \beta^2 \end{array}\right\} \quad (13)$$

Substituting $\alpha = 2pq,\ \beta = 3p^2 - q^2$ in (13) and (2), the corresponding integer solutions of (1) are

$$x(p,q) = -9p^4 - q^4 + 18p^2q^2 + 12p^3q - 4pq^3$$
$$y(p,q) = -9p^4 - q^4 + 18p^2q^2 - 12p^3q + 4pq^3$$

$$z(p,q) = -18p^4 - 2q^4 + 36p^2q^2$$

PROPERTIES:

- $x(1,q) + 24F_{4,q,3} + 20P_q^5 - T_{36,q} \equiv 13 \pmod{22}$
- $x(p,1) + 216F_{4,p,3} + 96P_p^5 + T_{90,p} \equiv -1 \pmod{107}$
- $y(1,q) + 24F_{4,q,3} + 4P_q^5 - T_{20,q} \equiv 1 \pmod{10}$
- $y(p,1) + 216F_{4,p,3} + 144P_p^5 + T_{42,p} \equiv -1 \pmod{75}$
- $z(1,q) + 48F_{4,q,3} + 36P_q^5 - T_{44,q} \equiv 0 \pmod{2}$

CONCLUSION:

In this paper we have analysed a diophantine equation of degree 6 with 4 unknowns for its non-zero distinct integer solutions. Also, we have presented relations between the solutions and special numbers. To conclude, one may search for other forms of sextic equations and analysed them for their corresponding distinct integer solutions.

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