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Research Article

Existence of Positive Solution for Nonlinear Third-Order Boundary Value Problem

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Abstract: The nonlinear third-order boundary value problem

$$u'''(t)-\rho^3u(t)=f(t,u(t)),0<t<1$$
$$u'(0)=0,u(1)=0,u''(1)=0$$

is studied in this work, where $0<\rho<\frac{4\pi}{3\sqrt{3}}$ .The existence result of at least one positive solution to above third-order boundary value problem is obtained by using fixed point index theory in cone.
Keywords: Third-order boundary value problem, Positive solution, Cone, Fixed point index

INTRODUCTION

In this paper, we are concerned with the following third-order nonlinear boundary value problem (BVP for short)

$$u'''(t)-\rho^3u(t)=f(t,u(t)),\ 0<t<1, \tag{1.1}$$
$$u'(0)=0,u(1)=0,u''(1)=0 \tag{1.2}$$

where $\rho\in(0,\frac{4\pi}{3\sqrt{3}})$ is a parameter, $f:[0,1]\times[0,+\infty)\to R$ is a nonnegative and continuous function. By a positive solution of BVP (1.1) and (1.2), we mean a function $u(t)$ which is positive on $(0,1)$ and $u(t)\in C^2[0,1]\cap C^3(0,1)$ such that $u(t)$ satisfied differential equation (1.1) and the boundary conditions (1.2).

It is assumed throughout that

(H_1) $f(t,u)$ is integral on $[0,1]$ for each fixed $u\in(0,+\infty)$ and $0<\int_0^1 f(t,u(t))\mathrm{d}t<+\infty$;

(H_2) $\limsup\limits_{u\to 0}\sup\limits_{t\in[0,1]}\frac{f(t,u)}{u}<\mu_1,\quad \liminf\limits_{u\to+\infty}\inf\limits_{t\in[0,1]}\frac{f(t,u)}{u}>\mu_2$;

(H_3) $\liminf\limits_{u\to 0}\inf\limits_{t\in[0,1]}\frac{f(t,u)}{u}>\mu_2,\quad \limsup\limits_{u\to+\infty}\sup\limits_{t\in[0,1]}\frac{f(t,u)}{u}<\mu_1$,

where $\mu_1=\frac{\pi^3}{8}-\rho^3,\mu_2=\frac{\pi^3e^{\rho}}{4(2-\sqrt{3})\sigma},\sigma=\frac{2\sqrt{3}\rho}{9\pi}e^{-\frac{3\rho}{2}}$.

Third-order differential equation arise from many branches of applied mathematics and physics, e.g., in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flow and so on [1]. In recent years, third-order BVP have been studied extensively by many researchers, see, for example, [2-9] discussed some third-order two-point BVP, while [10-18] studied some third-order three-point BVP. Utilizing the Leray-Schauder degree theory and lower and upper solution method, Du, Ge and Liu [3] investigated the existence of solution for the third-order nonlinear BVP

$$u'''(t)-f(t,u(t),u'(t),u''(t))=0,\ 0<t<1,$$

under two-point nonlinear boundary conditions

$$u(0)=0\,,\ g(u'(0),u''(0))=A,\ \ h(u'(1),u''(1))=B,$$

where $A,B\in R$, $f:[0,1]\times R^3\to R$ is continuous, $g,h:R^2\to R$ are continuous. Cabada [4] and Yao and Feng [5] used the method of lower and upper solution and fixed point theorems to establish the existence of periodic solutions and positive solutions for the third-order ordinary differential equation

$$u'''(t)-f(t,u(t))=0,\ a<t<b,$$

under two-point boundary conditions

$$u^{(i)}(0)=u^{(i)}(2\pi)\,,i=1,2,$$

with $a=0,b=2\pi$,and

$$u(0)=u'(0)=u'(1)=0$$

with $a=0,b=1$, respectively. Applying the Krasnosel'skii fixed point theorem, Li [2] and Sun [13] obtained existence of multiple positive solutions for the singular nonlinear third-order differential equations

$$u'''(t)-\lambda\alpha(t)f(t,u(t))=0\,,0<t<1,$$

under two-point boundary conditions

$$u(0)=u'(0)=u''(1)=0\,,$$

and three-point boundary conditions

$$u(0)=u'(\eta)=u''(1)=0,$$

respectively, where the function $\alpha(t)$ may be singular at $t=0,1$ and the function f has no singularity. Very recently, Liu, Ume, Anderson and Kang [6] have discussed the singular nonlinear third-order BVP

$$u'''(t)+\lambda\alpha(t)f(t,u(t))=0,\ \ a<t<b$$

$$u(a)=u''(a)=u'(b)=0$$

under the condition that $\lambda>0$ is a parameter, $\alpha\in C((a,b),R^+)$, $f\in C([a,b]\times(0,+\infty),R^+)$ and $\alpha(t)$ may be singular at $t=a,b$ and $f(t,u)$ may be singular at $u=0$, they established the existence of at least one or two non decreasing positive solutions by using the Green's function and the fixed-point theorem of cone expansion and compression type.

Inspired and motivated by the works mentioned above, In this work we will consider the existence of positive solution to the nonlinear BVP (1.1) and (1.2) however, to the author's knowledge, few papers in the literature can be found dealing with the existence of positive solution of (1.1) and(1.2). The purpose of this paper is to fill in the gap in this area. By means of the positivity of Green's function G *(t, s)* and the fixed point index theory in cone, we establish a few sufficient conditions for the existence of at least one positive solution to the BVP (1.1) and (1.2) if the nonlinearity *f* is either superlinear or sublinear. The results obtained extend and complement some known results.

The rest of the article is organized as follows. In Section 2, we present some preliminaries and the fixed point index theory in cone that will be used in Section 3.The main results and proofs will be given in Section 3.

Preliminaries and lemmas

Consider the Banach space $C[0,1]$ with norm $\|u\|=\max\limits_{0\le t\le 1}|u(t)|$ and let

$$C^+[0,1]=\left\{u\in C[0,1];u(t)\ge 0,0\le t\le 1\right\},$$

$$K=\{v(t)\in C^+[0,1];\ \min_{t\in[\frac{1}{3},\frac{2}{3}]} v(t)\ge\sigma\|v\|,0\le t\le 1\},$$

it is easy to check that K is a cone of nonnegative function in $C[0,1]$.Define the operator $J:C^+[0,1]\to C^+[0,1]$ as follows

$$Jv(t)=\int_t^1 e^{\rho(t-s)}v(s)\mathrm{d}s$$

Consider the nonlinear second order boundary problem

$$v''(t)+\rho v'(t)+\rho^2 v(t)=-f(t,Jv(t)) \qquad (2.1)$$

$$v(0)=\rho\int_0^1 e^{-\rho s}v(s)ds, \quad v'(1)+\rho v(1)=0 \qquad (2.2)$$

A direct check implies that the problem (2.1), (2.2) is equivalent to the following integral equation

$$v(t)=\int_0^1 G(t,s)f(s,Jv(s))\mathrm{d}s+\int_0^1 \frac{\rho e^{-\frac{\rho}{2}(t+2s)}\sin[\frac{\sqrt{3}}{2}\rho(1-t)+\frac{\pi}{3}]}{\sin(\frac{\sqrt{3}}{2}\rho+\frac{\pi}{3})}v(s)ds \qquad (2.3)$$

where

$$G(t,s)=\begin{cases}\dfrac{2e^{-\frac{\rho}{2}(t-s)}\sin\frac{\sqrt{3}}{2}\rho s\sin[\frac{\sqrt{3}}{2}\rho(1-t)+\frac{\pi}{3}]}{\sqrt{3}\rho\sin(\frac{\sqrt{3}}{2}\rho+\frac{\pi}{3})}, 0\le s\le t\le 1\\[2ex] \dfrac{2e^{-\frac{\rho}{2}(t-s)}\sin\frac{\sqrt{3}}{2}\rho t\sin[\frac{\sqrt{3}}{2}\rho(1-s)+\frac{\pi}{3}]}{\sqrt{3}\rho\sin(\frac{\sqrt{3}}{2}\rho+\frac{\pi}{3})}, 0\le t\le s\le 1\end{cases} \qquad (2.4)$$

Clearly, if $v(t)$ is a positive solution of the problem (2.1) and (2.2), and let $u(t)=Jv(t)$, then it is easy to know $u(t)$ is a positive solution of the problem (1.1), (1.2).

Lemma 2.1: For all $(s,t)\in[0,1]\times[0,1]$,we have

$$\frac{\sqrt{3}\rho}{\pi}e^{-\frac{\rho}{2}}t(1-t)G(s,s)\le G(t,s)\le G(s,s)$$

Proof: Since $\sin x\ge\frac{2}{\pi}x$ for $x\in[0,\frac{\pi}{2}]$ and $\sin x\ge 2(1-\frac{x}{\pi})$ for $x\in[\frac{\pi}{2},\pi]$,we have $\sin x\ge\frac{2}{\pi}x(1-\frac{x}{\pi})$ for $x\in[0,\pi]$.It follows from $0<\rho<\frac{4\pi}{3\sqrt{3}}$ that

$$\sin\frac{\sqrt{3}}{2}\rho t\ge\frac{\sqrt{3}}{\pi}\rho t(1-\frac{\sqrt{3}}{2\pi}\rho t)>\frac{\sqrt{3}}{\pi}\rho t(1-t),$$

$$\sin[\frac{\sqrt{3}}{2}\rho(1-t)+\frac{\pi}{3}]\ge\frac{2}{\pi}[\frac{\sqrt{3}}{2}\rho(1-t)+\frac{\pi}{3}][(1-\frac{1}{\pi}(\frac{\sqrt{3}}{2}\rho(1-t)+\frac{\pi}{3})]>\frac{\sqrt{3}}{\pi}\rho t(1-t)$$

and

$$\frac{G(t,s)}{G(s,s)}=\begin{cases}\dfrac{e^{-\frac{\rho}{2}(t-s)}\sin[\frac{\sqrt{3}}{2}\rho(1-t)+\frac{\pi}{3}]}{\sin[\frac{\sqrt{3}}{2}\rho(1-s)+\frac{\pi}{3}]}, 0\le s\le t\le 1\\[2ex] \dfrac{e^{-\frac{\rho}{2}(t-s)}\sin\frac{\sqrt{3}}{2}\rho t}{\sin\frac{\sqrt{3}}{2}\rho s}, 0\le t\le s\le 1\end{cases}$$

$$\ge\frac{\sqrt{3}\rho}{\pi}e^{-\frac{\rho}{2}}t(1-t)$$

It is obvious that $G(t,s) \le G(s,s)$.The proof is complete.

Define an integral operator $\Phi: C^+[0,1] \to C^+[0,1]$ by

$$\Phi v(t) = \int_0^1 G(t,s) f(s, Jv(s))\mathrm{d}s + \int_0^1 \frac{\rho e^{-\frac{\rho}{2}(t+2s)} \sin[\frac{\sqrt{3}}{2}\rho(1-t) + \frac{\pi}{3}]}{\sin(\frac{\sqrt{3}}{2}\rho + \frac{\pi}{3})} v(s)ds \qquad (2.5)$$

Then, only if only the nonzero fixed point $v(t)$ of mapping Φ defined by (2.5) is a positive solution of (2.1) and (2.2).

Lemma 2.2: $\Phi(K) \subset K$

Proof: For any $v \in K$, from lemma 2.1 we have inequalities

$$\| \Phi v \| \le \int_0^1 G(s,s) f(s, Jv(s))\mathrm{d}s + \int_0^1 \frac{\rho v(s)}{\sin(\frac{\sqrt{3}}{2}\rho + \frac{\pi}{3})} ds$$

and

$$\min_{t\in[\frac{1}{3},\frac{2}{3}]} \Phi v(t) \ge \min_{t\in[\frac{1}{3},\frac{2}{3}]} \frac{\sqrt{3}\rho}{\pi} t(1-t)[e^{-\frac{\rho}{2}} \int_0^1 G(s,s) f(s, Jv(s))\mathrm{d}s + e^{-\frac{3\rho}{2}} \int_0^1 \frac{\rho v(s)}{\sin(\frac{\sqrt{3}}{2}\rho + \frac{\pi}{3})} ds]$$

$$\ge \frac{2\sqrt{3}\rho}{9\pi} e^{-\frac{3\rho}{2}} [\int_0^1 G(s,s) f(s, Jv(s))\mathrm{d}s + \int_0^1 \frac{\rho v(s)}{\sin(\frac{\sqrt{3}}{2}\rho + \frac{\pi}{3})} ds] \ge \sigma \| \Phi v \|$$

Thus, $\Phi(K) \subset K$

It is clear that $\Phi: K \to K$ is a completely continuous mapping.

Let E be a Banach space and $K \subset E$ be a cone in E.Assume that Ω is a bounded open subset of E and let $\partial\Omega$ be a its boundary. Let $\Phi: K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If $\Phi u \ne u$ for every $u \in K \cap \partial\Omega$, then the fixed point index $i(\Phi, K \cap \Omega, K)$ is defined. If $i(\Phi, K \cap \Omega, K) \ne 0$, then Φ has a fixed point in $K \cap \Omega$.

For $r > 0$, let $K_r = \{u \in K; \| u \| < r\}$ and $\partial K_r = \{u \in K; \| u \| = r\}$,which is the boundary of K_r in K.The following two lemmas are needed in our argument.

Lemma 2.3 [20]: Let $\Phi: K \to K$ be a completely continuous mapping with $\lambda\Phi u \ne u$ for every $u \in \partial K_r$ and $0 < \lambda \le 1$.Then $i(\Phi, K_r, K) = 1$.

Lemma 2.4 [20]: Let $\Phi: K \to K$ be a completely continuous mapping. Suppose that the following two condition are satisfied: (i) $\inf_{u\in\partial K_r} \| Tu \| > 0$; (ii) $\lambda\Phi u \ne u$ for every $u \in \partial K_r$ and $\lambda \ge 1$.Then $i(\Phi, K_r, K) = 0$.

RESULTS

Theorem 3.1: Assume that (H_1) and (H_2) hold, then the problem (1.1) and (1.2) has at least one positive solution.

Proof: Now we show that Φ has a nonzero fixed point in the cases (H_1) and (H_2). By (H_2) we may choose $\varepsilon \in (0, \mu_1)$ and $r > 0$ so that

$$f(t,u) \le (\mu_1 - \varepsilon)u, \ \forall t \in [0,1], 0 \le u \le r.$$

Let $K_r = \{v \in K; \| v \| < r\}$,we now prove that $\lambda\Phi v \ne v$ for any $v \in \partial K_r$ and $0 < \lambda \le 1$. In fact, if there exist $v_0 \in \partial K_r$ and $0 < \lambda_0 \le 1$ such that $\lambda_0 \Phi v_0 = v_0$, then by definition of Φ, $v_0(t)$ satisfies differential equation

$$v_0''(t)+\rho v_0'(t)+\rho^2 v_0(t)=-\lambda_0 f(t,Jv_0(t)) \qquad (3.1)$$

and boundary condition

$$v_0(0)=\rho\int_0^1 e^{-\rho s}v_0(s)ds, \quad v_0'(1)+\rho v_0(1)=0 \qquad (3.2)$$

Let $u_0(t)=Jv_0(t)=\int_t^1 e^{\rho(t-s)}v_0(s)\mathrm{d}s$, then $0<u_0(t)\le\|v_0\|=r$,and $u_0(t)$ satisfies

$$u_0'''(t)-\rho^3 u_0(t)=\lambda_0 f(t,u_0(t)) \qquad (3.3)$$

$$u_0'(0)=0, u_0(1)=0, u_0''(1)=0 \qquad (3.4)$$

Multiplying Eq. (3.3) by $\sin\frac{\pi}{2}t$ and integrating on $[0,1]$,by using (3.4) we have

$$\frac{\pi^3}{8}\int_0^1 u_0(t)\cos\frac{\pi}{2}t\mathrm{d}t-\rho^3\int_0^1 u_0(t)\sin\frac{\pi}{2}t\mathrm{d}t=\lambda_0\int_0^1 f(t,u_0(t))\sin\frac{\pi}{2}t\mathrm{d}t \qquad \le(\mu_1-\varepsilon)\int_0^1 u_0(t)\sin\frac{\pi}{2}t\mathrm{d}t \qquad (3.5)$$

and hence

$$\frac{\pi^3}{8}\int_0^1 u_0(t)(\cos\frac{\pi}{2}t-\sin\frac{\pi}{2}t)\mathrm{d}t\le-\varepsilon\int_0^1 u_0(t)\sin\frac{\pi}{2}t\mathrm{d}t<0 \qquad (3.6)$$

since $u_0'''(t)=\rho^3 u_0(t)+\lambda_0 f(t,u_0(t))>0$, we know that $u''(t)$ is increasing and $u_0''(t)<u_0''(1)=0$,and hence $u_0'(t)$ is decreasing and $u_0'(t)<u_0'(0)=0$,thus, $u_0(t)$ is decreasing and $u_0(t)>u_0(1)=0$ .By using integral mean value theorem we have

$$\int_0^1 u_0(t)(\cos\frac{\pi}{2}t-\sin\frac{\pi}{2}t)\mathrm{d}t=\int_0^{\frac{1}{2}} u_0(t)(\cos\frac{\pi}{2}t-\sin\frac{\pi}{2}t)\mathrm{d}t+\int_{\frac{1}{2}}^1 u_0(t)(\cos\frac{\pi}{2}t-\sin\frac{\pi}{2}t)\mathrm{d}t$$

$$=u_0(\xi_1)\int_0^{\frac{1}{2}}(\cos\frac{\pi}{2}t-\sin\frac{\pi}{2}t)\mathrm{d}t+u_0(\xi_2)\int_{\frac{1}{2}}^1(\cos\frac{\pi}{2}t-\sin\frac{\pi}{2}t)\mathrm{d}t$$

$$=\frac{2}{\pi}(\sqrt{2}-1)[u_0(\xi_1)-u_0(\xi_2)]\ge 0 \qquad (3.7)$$

where $0\le\xi_1\le\xi_2\le 1$,which contradicts (3.6) . By Lemma 2.3 we have

$$i(\Phi,K_r,K)=1. \qquad (3.8)$$

It follows from (H_2) that there exist $\varepsilon>0$ and $H>0$ such that

$$f(t,u)\ge(\mu_2+\varepsilon)u, \ \forall t\in[0,1], u\ge H$$

set $c=\max_{\substack{t\in[0,1]\\ u\in[0,H]}}|f(t,u)-(\mu_2+\varepsilon)u|$, then it is clear to see

$$f(t,u)\ge(\mu_2+\varepsilon)u-c\,.\,\forall t\in[0,1], u>0\,.$$

Choose $R>\max\{r,H,\frac{\pi c e^{\rho}}{(2-\sqrt{3})\sigma\varepsilon},\frac{3ce^{\rho}}{\rho\sigma}\int_0^1 G(s,s)ds\}$ and let $v\in\partial K_R$ .Since $Jv(t)=\int_t^1 e^{\rho(t-s)}v(s)\mathrm{d}s\ge 0$,we see that $f(t,Jv(t))\ge(\mu_2+\varepsilon)Jv(t)-c$ .From Lemma2.1 we obtain

$$\|\Phi v\|\ge\max_{t\in[0,1]}\frac{\sqrt{3}\rho}{\pi}t(1-t)\{e^{-\frac{\rho}{2}}\int_0^1 G(s,s)[(\mu_2+\varepsilon)Jv(s)-c]\mathrm{d}s+\frac{\rho e^{-\frac{3\rho}{2}}}{\sin(\frac{\sqrt{3}}{2}\rho+\frac{\pi}{3})}\int_0^1 v(s)ds\}$$

$$\ge\frac{\sqrt{3}\rho}{4\pi}e^{-\frac{3\rho}{2}}[\rho\int_{\frac{1}{3}}^{\frac{2}{3}}v(s)ds-ce^{\rho}\int_0^1 G(s,s)ds]$$

$$\geq \frac{\sqrt{3}\rho}{12\pi} e^{-\frac{3\rho}{2}} [\rho\sigma \| v \| - 3ce^{\rho} \int_0^1 G(s,s) ds] > 0 \qquad (3.9)$$

Therefore $\inf_{v \in \partial K_R} \| \Phi v \| > 0$.

Next we show that $\lambda \Phi v \neq v$ for any $v \in \partial K_R$ and $\lambda \geq 1$.In fact , if there exist $v_0 \in \partial K_R$ and $\lambda_0 \geq 1$ such that $\lambda_0 \Phi v_0 = v_0$ then $v_0(t)$ satisfies Eq. (3.1) and boundary condition (3.2). Let $u_0(t) = Jv_0(t) = \int_t^1 e^{\rho(t-s)} v_0(s) \mathrm{d}s$, then $u_0(t)$ satisfies Eq. (3.3) and boundary condition (3.4). As the calculation as that of (3.5), we obtain

$$\frac{\pi^3}{8} \int_0^1 u_0(t) \cos\frac{\pi}{2} t \mathrm{d}t - \rho^3 \int_0^1 u_0(t) \sin\frac{\pi}{2} t \mathrm{d}t = \lambda_0 \int_0^1 f(t, u_0(t)) \sin\frac{\pi}{2} t \mathrm{d}t$$

$$\geq (\mu_2 + \varepsilon) \int_0^1 Jv_0(t) \sin\frac{\pi}{2} t \mathrm{d}t - \int_0^1 c \sin\frac{\pi}{2} t \mathrm{d}t$$

$$\geq \frac{2}{\pi} e^{-\rho} (\mu_2 + \varepsilon) \int_0^1 (1 - \cos\frac{\pi}{2} t) v_0(t) \mathrm{d}t - \frac{2}{\pi} c$$

$$\geq \frac{2}{\pi} e^{-\rho} (\mu_2 + \varepsilon) \sigma \| v_0 \| \int_{\frac{1}{3}}^{\frac{2}{3}} (1 - \cos\frac{\pi}{2} t) \mathrm{d}t - \frac{2}{\pi} c$$

$$> \frac{2(2-\sqrt{3})\sigma}{\pi^2} e^{-\rho} (\mu_2 + \varepsilon) \| v_0 \| - \frac{2c}{\pi} \qquad (3.10)$$

since

$$\frac{\pi^3}{8} \int_0^1 u_0(t) \cos\frac{\pi}{2} t \mathrm{d}t = \frac{\pi^3}{8} \int_0^1 \cos\frac{\pi}{2} t \mathrm{d}t \int_t^1 e^{\rho(t-s)} v_0(s) ds$$

$$\leq \frac{\pi^3}{8} \| v_0 \| \int_0^1 (1-t) \cos\frac{\pi}{2} t dt = \frac{\pi}{2} \| v_0 \|. \qquad (3.11)$$

Thus , by (3.10) and (3.11) we have

$$\frac{\pi}{2} \| v_0 \| \geq \frac{2(2-\sqrt{3})}{\pi^2} \sigma e^{-\rho} (\mu_2 + \varepsilon) \| v_0 \| - \frac{2}{\pi} c, \qquad (3.12)$$

therefore $\| v_0 \| \leq \frac{\pi c e^{\rho}}{(2-\sqrt{3})\sigma\varepsilon} < R$, this is in contradiction with $\| v_0 \| = R$. By Lemma 2.3 we know that

$$i(\Phi, K_R, K) = 0. \qquad (3.13)$$

Now by the additivity of fixed point index, (3.8) and (3.13) we have

$$i(\Phi, K_R \setminus \overline{K_r}, K) = i(\Phi, K_R, K) - i(\Phi, K_r, K) = -1.$$

Therefore Φ has a fixed point $v(t)$ in $K_R \setminus \overline{K_r}$, which is the positive solution of the BVP (2.1) and (2.2),and satisfied $r < \| v \| < R$.Let $u(t) = Jv(t) = \int_t^1 e^{\rho(t-s)} v(s) \mathrm{d}s$,then $u(t)$ is the positive solution of the BVP (1.1) and (1.2).

Theorem 3.2: Assume that (H_1) and (H_3) hold, then the problem (1.1) and (1.2) has at least one positive solution.

Proof: By (H_3), there exist $\varepsilon > 0$ and $r > 0$ such that

$$f(t,u) \geq (\mu_2 + \varepsilon) u, \forall t \in [0,1], 0 \leq u \leq r.$$

Then for every $v \in \partial K_r$, from $Jv(t) = \int_t^1 e^{\rho(t-s)} v(s) \mathrm{d}s \leq \| v \| = r$,we have

$$f(t, Jv) \geq (\mu_2 + \varepsilon) Jv,$$

thus, the calculation is similar to (3.9) ,we obtain

$$\|\Phi v\| \geq \max_{t\in[0,1]} \frac{\sqrt{3}\rho}{\pi} t(1-t)[e^{-\frac{\rho}{2}}\int_0^1 G(s,s)(\mu_2+\varepsilon)Jv(s)\mathrm{d}s + \frac{\rho e^{-\frac{3\rho}{2}}}{\sin(\frac{\sqrt{3}}{2}\rho+\frac{\pi}{3})}\int_{\frac{1}{3}}^{\frac{2}{3}} v(s)ds]$$

$$\geq \frac{\sqrt{3}\rho^2\sigma r}{12\pi} e^{-\frac{3\rho}{2}}$$

Hence $\inf_{v\in\partial K_R} \|\Phi v\| > 0$.

Next, we show that $\lambda\Phi v \neq v$ for any $v\in\partial K_r$ and $\lambda \geq 1$.In fact, if there exist $v_0\in\partial K_r$ and $\lambda_0\geq 1$ such that $\lambda_0\Phi v_0 = v_0$, then by definition of Φ, $v_0(t)$ satisfies differential equation (3.1) and boundary condition (3.2).Let $u_0(t)=Jv_0(t)$,then $u_0(t)$ satisfies (3.3) and (3.4). The calculation is similar to (3.5), we obtain

$$\frac{\pi^3}{8}\int_0^1 u_0(t)\cos\frac{\pi}{2}t\mathrm{d}t - \rho^3\int_0^1 u_0(t)\sin\frac{\pi}{2}t\mathrm{d}t = \lambda_0\int_0^1 f(t,u_0(t))\sin\frac{\pi}{2}t\mathrm{d}t \qquad (3.14)$$

As the calculation as that of (3.10) and (3.11), we have

$$\int_0^1 f(t,Jv_0(t))\sin\frac{\pi}{2}t\mathrm{d}t \geq \frac{2(2-\sqrt{3})}{\pi^2}\sigma e^{-\rho}(\mu_2+\varepsilon)\|v_0\|$$

and

$$\frac{\pi^3}{8}\int_0^1 u_0(t)\cos\frac{\pi}{2}t\mathrm{d}t \leq \frac{\pi}{2}\|v_0\|.$$

And hence from (3.14) we know that

$$\frac{\pi}{2}\|v_0\| > \frac{2(2-\sqrt{3})}{\pi^2}\sigma e^{-\rho}(\mu_2+\varepsilon)\|v_0\|$$

therefore $\varepsilon(2-\sqrt{3})\sigma e^{-\rho} < 0$,which is a contradiction. Thus, by Lemma 2.4 we have

$$i(\Phi, K_r, K) = 0. \qquad (3.15)$$

Again by (H_3), we know that there exist $\varepsilon\in(0,\mu_1)$ and $H>0$ such that

$$f(t,u) \leq (\mu_1-\varepsilon)u,\ \ \forall t\in[0,1],\ u\geq H.$$

Set $c = \max_{\substack{t\in[0,1]\\ u\in[0,H]}} |f(t,u)-(\mu_1-\varepsilon)u|$, it is clear that

$$f(t,u) \leq (\mu_1-\varepsilon)u + c,\ \forall t\in[0,1],\ u\geq 0$$

Choose $R > \max\{r, H, \frac{\pi c e^{\rho}}{(2-\sqrt{3})\sigma\varepsilon}\}$, next, we show that $\lambda\Phi v\neq v$ for any $v\in\partial K_r$ and $0<\lambda\leq 1$. In fact, if there exist $v_0\in\partial K_R$ and $0<\lambda_0\leq 1$ such that $\lambda_0\Phi v_0 = v_0$, then, as the calculation as that of (3.11) we obtain

$$\frac{\pi^3}{8}\int_0^1 u_0(t)\cos\frac{\pi}{2}t\mathrm{d}t - \rho^3\int_0^1 u_0(t)\sin\frac{\pi}{2}t\mathrm{d}t \leq (\mu_1-\varepsilon)\int_0^1 u_0(t)\sin\frac{\pi}{2}t\mathrm{d}t + \frac{2c}{\pi}$$

and hence

$$\frac{\pi^3}{8}\int_0^1 u_0(t)(\cos\frac{\pi}{2}t - \sin\frac{\pi}{2}t)\mathrm{d}t \leq -\varepsilon\int_0^1 u_0(t)\sin\frac{\pi}{2}t\mathrm{d}t + \frac{2c}{\pi} \qquad (3.16)$$

As the calculation as that of (3.7) ,we have

$$\int_0^1 u_0(t)(\cos\frac{\pi}{2}t - \sin\frac{\pi}{2}t)\mathrm{d}t \geq 0,$$

and hence

$$\int_0^1 u_0(t)\sin\frac{\pi}{2}t\mathrm{d}t \le \frac{2c}{\pi\varepsilon}.$$

On the other hand, we have that

$$\int_0^1 u_0(t)\sin\frac{\pi}{2}t\mathrm{d}t = \int_0^1 \sin\frac{\pi}{2}t\mathrm{d}t\int_t^1 e^{\rho(t-s)}v(s)ds \ge \frac{2(2-\sqrt{3})\sigma}{\pi^2}e^{-\rho}\,\|v_0\|,$$

thus, from (3.16) we know that

$$\frac{\pi(\rho+2e^{-\rho})}{4+\pi\rho^2}\sigma\,\|v_0\|\le\frac{2c}{\pi\varepsilon},$$

Therefore, $\|v_0\|\le\dfrac{\pi c e^{\rho}}{(2-\sqrt{3})\sigma\varepsilon}<R$, this is in contradiction with $\|v_0\|=R$. Hence by Lemma 2.3 we have

$$i(\Phi,K_R,K)=1. \qquad (3.17)$$

It follows from (3.15) and (3.17) that we have

$$i(\Phi,K_R\setminus\overline{K}_r,K)=i(\Phi,K_R,K)-i(\Phi,K_r,K)=1$$

Therefore Φ has a fixed point $v(t)$ in $K_R\setminus\overline{K}_r$, which is the positive solution of the BVP (2.1) and (2.2),and satisfied $r<\|v\|<R$.Let $u(t)=Jv(t)=\int_t^1 e^{\rho(t-s)}v(s)\mathrm{d}s$,then $u(t)$ is the positive solution of the BVP (1.1) and (1.2).

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