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Research Article

Existence of Positive Solutions for Nonlinear Fourth-order Boundary Value Problems

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Abstract: The nonlinear fourth-order boundary value problem

$$u^{(4)}(t)-\rho^4u(t)=f(t,u(t))$$
$$u(0)=u'(1)=u''(0)=u'''(1)=0$$

is studied in this paper, where $0<\rho<\frac{\pi}{2}$.The existence result of at least one positive solution to above fourth-order boundary value problem is obtained by using Fixed Point theorem in cones.
Keywords: Fourth-order boundary value problem, Cone, Positive solutions, fixed point
MSC: 34B10, 34B15

INTRODUCTION

In this paper, we consider the nonlinear fourth-order boundary value problems (BVP for short)

$$u^{(4)}(t)-\rho^4u(t)=f(t,u(t)),0<t<1, \tag{1.1}$$

where $0<\rho<\frac{\pi}{2}$ is a parameter , $f:[0,1]\times[0,+\infty)\to R$ is a nonnegative and continuous function. With boundary conditions

$$u(0)=u'(1)=u''(0)=u'''(1)=0 \tag{1.2}$$

Describing, In this case, a beam deformation with one endpoint simply supported and the other one sliding clamped. By a positive solution of BVP (1.1) and (1.2) ,we call a function $u(t)$ which is positive on $(0,1)$ and $u(t)\in C^3[0,1]\cap C^4[0,1]$ such that $u(t)$ satisfied differential equation (1.1) and the boundary conditions (1.2) .It is assumed throughout that

(H_1): $f(t,u)$ is integral for each fixed u$\in[0,1]\times[0,+\infty)$, and $0<\int_0^1 f(t,u(t))\mathrm{d}t<+\infty$;

(H_2): $\lim\limits_{u\to0^+}\sup\limits_{t\in[0,1]}\frac{f(t,u)}{u}=0,\lim\limits_{u\to\infty}\inf\limits_{t\in[0,1]}\frac{f(t,u)}{u}=\infty$; $f(t,u)$ for super-linear

(H_3): $\lim\limits_{u\to0^+}\sup\limits_{t\in[0,1]}\frac{f(t,u)}{u}=\infty,\lim\limits_{u\to\infty}\inf\limits_{t\in[0,1]}\frac{f(t,u)}{u}=0.$ $f(t,u)$ for sub-linear

The nonlinear fourth-order equations appear in some physical problems as, for example, the bending of an elastic beam with several types of two point boundary conditions, describing how the beam is supported at the two endpoints, see[10,11,13] .The positive solution has profound practical significance .Many scholars had research and got a lot of good results, see[4-9].Using an a priori estimation ,lower and upper solution method and degree theory, T. Gyulov investigated the existence of solution for a fully nonlinear beam equation

$$u^{(iv)} = f\left(t,u,u',u'',u'''\right), 0<t<1$$

with boundary conditions

$$u(0)=u'(1)=u''(0)=u'''(1)=0$$

where $f:[0,1]\times R^4 \to R$ is a continuous function satisfying a Nagumo-type condition. This result deals with a fully nonlinear differential equation-the nonlinearity can depend on every derivative until order three.

In 1996, Dalmasso first proved the existence of single positive solution of problem

$$u^{(4)}(t)=h(t)\,f\left(t,u(t),u'(t)\right), t\in[0,1]\setminus E$$

under two -point boundary conditions

$$u(0)=u'(0)=u''(1)=u'''(1)=0$$

By comparing the first value of associated linear problem with the limits

$$\lim\sup_{u\to 0}\frac{f(u)}{u} \quad \lim\inf_{u\to\infty}\frac{f(u)}{u}$$

When $E\neq\phi, h(t)\equiv 1, f(t,u)=f(u)$ and $f:[0,+\infty]\to[0,+\infty]$ is continuous.

For boundary conditions considering every derivative until order there ,similar results can be obtained, since the second and third derivative are given in different endpoints. More precisely, considering Eq.(1.1) with one of the following boundary conditions

$$u(0)=u'(1)=u''(1)=u'''(0)=0$$

$$u(1)=u'(0)=u''(0)=u'''(1)=0$$

$$u(1)=u'(0)=u''(1)=u'''(0)=0$$

analogous theorems hold, with adequate modifications, assuming the second derivatives of lower and upper solutions in the reversed order. For boundary value problems including Eq.(1.1) and one of the following conditions

$$u(0)=u'(0)=u''(0)=u'''(1)=0$$

$$u(0)=u'(1)=u''(1)=u'''(0)=0$$

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$$u(1)=u'(1)=u''(1)=u'''(1)=0$$

existence and location results are obtained for another type of lower and upper solution, with the second derivatives well ordered.

Inspired and motivated by the works mentioned above, in this paper, we will consider the existence of positive solution to the nonlinear BVP (1.1) and (1.2). The purpose of this paper is to fill in the gap in this area. The results obtained extend and complement some known results.

The rest of the article is organized as follows, In Section 2,we present some preliminaries and the fixed point theory in cone that willed be used in Section 3.The main results and proofs will be given in Section 3.

PRELIMINARIES AND LEMMAS

Consider the Banach space $C[0,1]$ with norm $\|u\|=\max_{0\le t\le 1}|u(t)|$ and let $C^+[0,1]=\{u\in C[0,1];u(t)\ge 0,0\le t\le 1\}$, $K=\left\{u(t)\in C^+[0,1];\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]}u(t)\ge\sigma\|u\|,0\le t\le 1\right\}$,

$$\sigma=\frac{3\rho}{16}\operatorname{csch}(\rho)$$

it is easy to check that K is a cone of nonnegative function in $C[0,1]$.

Consider the nonlinear second order boundary problem

$$u''(t)-\rho^2u(t)=-v(t) \tag{2.1}$$

$$u(0)=0,u'(1)=0 \tag{2.2}$$

A direct check implies that the problem (2.1), (2.2) is equivalent to the following integral equation

$$u(t)=\int_0^1 G_1(t,s)v(s)\mathrm{d}s, \tag{2.3}$$

Where

$$G_1(t,s)=\begin{cases}\dfrac{\sinh(\rho s)\cosh(\rho-\rho t)}{\rho\cosh(\rho)},0\le s\le t\le 1\\ \dfrac{\sinh(\rho t)\cosh(\rho-\rho s)}{\rho\cosh(\rho)},0\le t\le s\le 1\end{cases} \tag{2.4}$$

Consider the nonlinear second order boundary problem

$$v''(t)+\rho^2v(t)=-f(t,u(t)) \tag{2.5}$$

$$v(0)=0,v'(1)=0 \tag{2.6}$$

The same method

$$v(t)=\int_0^1 G_2(t,s)f(s,u(s))\mathrm{d}s \tag{2.7}$$

where

$$G_2(t,s)=\begin{cases}\dfrac{\cos\rho(1-t)\sin\rho s}{\rho\cos\rho},0\le s\le t\le 1\\ \dfrac{\cos\rho(1-s)\sin\rho t}{\rho\cos\rho},0\le t\le s\le 1\end{cases} \tag{2.8}$$

we can easies compute $u(t)=\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\mathrm{d}s$

Clearly, if $u(t)$ is a positive solution of the problem (2.1) and (2.2) and let $u(t)=\Phi u(t)$, it is easy to know $u(t)$ is the positive solution of the BVP (1.1), (1.2).

Lemma 2.1 Foe all $(s,t)\in[0,1]\times[0,1]$, we have

$$\frac{G_1(t,s)}{G_1(s,s)}=\begin{cases}\dfrac{\cosh(\rho-\rho t)}{\cosh(\rho-\rho s)},0\le s\le t\le 1\\ \dfrac{\sinh(\rho t)}{\sinh(\rho s)},0\le t\le s\le 1\end{cases}$$

$$\rho t(1-t)\operatorname{csch}(\rho)G_1(s,s)\le G_1(t,s)\le G_1(s,s)$$

Proof. It is clearly to see

$$\frac{G_1(t,s)}{G_1(s,s)} = \begin{cases} \dfrac{\cosh(\rho-\rho t)}{\cosh(\rho-\rho s)}, 0 \le s \le t \le 1 \\ \dfrac{\sinh(\rho t)}{\sinh(\rho s)}, 0 \le t \le s \le 1 \end{cases}$$

$$\ge \begin{cases} \rho t(1-t)\operatorname{sech}(\rho), 0 \le s \le t \le 1 \\ \rho t \operatorname{csch}(\rho), 0 \le t \le s \le 1 \end{cases}$$

$$\ge \rho t(1-t)\operatorname{csch}(\rho)$$

It is obvious that $G_1(t,s) \le G_1(s,s)$.The proof is complete.

Define an integral operator $\Phi : C^+[0,1] \to C^+[0,1]$ by

$$\Phi u(t) = \int_0^1 \int_0^1 G_1(t,s) G_2(s,\tau) f(\tau, u(\tau)) \mathrm{d}\tau \mathrm{d}s \tag{2.9}$$

Then, only if nonzero fixed point $u(t)$ of mapping Φ defined by (2.9) is a positive solution of (1.1) and (1.2)

Lemma 2.2 $\Phi(K) \subset K$

Proof. For any $u \in K$, from lemma 2.1we have

$$\|\Phi u(t)\| = \max \int_0^1 \int_0^1 G_1(t,s) G_2(s,\tau) f(\tau, u(\tau)) \mathrm{d}\tau \mathrm{d}s$$

And inequalities

$$\|\Phi u(t)\| \le \int_0^1 \int_0^1 G_1(s,s) G_2(s,\tau) f(\tau, u(\tau)) \mathrm{d}\tau \mathrm{d}s$$

$$\min_{t \in \left[\frac{1}{4},\frac{3}{4}\right]} \Phi u(t) = \min_{t \in \left[\frac{1}{4},\frac{3}{4}\right]} \int_0^1 \int_0^1 G_1(t,s) G_2(s,\tau) f(\tau, u(\tau)) \mathrm{d}\tau \mathrm{d}s$$

$$\ge \rho t(1-t)\operatorname{csch}(\rho) \int_0^1 \int_0^1 G_1(s,s) G_2(s,\tau) f(\tau, u(\tau)) \mathrm{d}\tau \mathrm{d}s$$

$$\ge \sigma \int_0^1 \int_0^1 G_1(s,s) G_2(s,\tau) f(\tau, u(\tau)) \mathrm{d}\tau \mathrm{d}s$$

$$\ge \sigma \|\Phi u(t)\| = \sigma \|\Phi u\|$$

Thus, $\Phi(k) \subset K$

It is clear that $\Phi : K \to K$ is a completely continuous mapping.

Let E be a Banach space, and let $K \subset E$ be a cone in E.Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$ and let $\Phi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either

(1) $\| \Phi u \| \le \| u \|$, $u \in K \cap \partial\Omega_1$, and $\| \Phi u \| \ge \| u \|$, $u \in K \cap \partial\Omega_2$;or

(2) $\| \Phi u \| \ge \| u \|$, $u \in K \cap \partial\Omega_1$, and $\| \Phi u \| \le \| u \|$, $u \in K \cap \partial\Omega_2$,

Then Φ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

We will apply the first and second parts of the above Fixed Point Theorem to the super-linear and sub-linear cases.

MAIN RESULTS

Theorem 3.1 Assume that $(H_1),(H_2)$ hold ,then the problem (1.1) and (1.2) has at least one positive solution.

Proof Since (H_2) ,we may choose $r > 0$ so that $f(t,u) \le \varepsilon u$,for $0 \le u \le r$, where $\varepsilon > 0$ satisfies

$$\varepsilon\int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)f\left(\tau,u\left(\tau\right)\right)\mathrm{d}\tau\mathrm{d}s\le 1.$$

Let $\Omega_1=\{u\in C[0,1];\|u\|<r\}$, $\forall u\in K\bigcap\partial\Omega_1$ from lemma 2.1,we have

$$\left\|\Phi u\left(t\right)\right\|\le\int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)f\left(\tau,u\left(\tau\right)\right)\mathrm{d}\tau\mathrm{d}s$$

$$\le\varepsilon\left\|u\right\|\int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s$$

$$\le\left\|u\right\|$$

Then shows $\|\Phi u\|\le\|u\|$.

Further, since (H_2) there exists $R_1>0$ such that $f(t,u)\ge\mu u$, $u\ge R_1$ where $\mu>0$ chosen so that

$$\mu\sigma\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s\ge 1$$

Let $R>\max\{r,\dfrac{R_1}{\sigma}\}$ and $\Omega_2=\{u\in C[0,1];\|u\|<R\}$, then $\forall u\in K\bigcap\partial\Omega_2$ and $\min\limits_{t\in[1/4,3/4]} u(t)\ge\sigma\|u\|=\sigma R>R_1$, implies

$$\left\|\Phi u\left(t\right)\right\|\ge\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]}\sigma\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,s)G_2(s,\tau)f\left(\tau,u\left(\tau\right)\right)\mathrm{d}\tau\mathrm{d}s\quad\ge\sigma\mu\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,s)G_2(s,\tau)u\left(\tau\right)\mathrm{d}\tau\mathrm{d}s$$

$$\ge\left\|u\right\|$$

Hence, $\|\Phi u\|\ge\|u\|$ for $\forall u\in K\bigcap\partial\Omega_2$

Therefore , by the first part of the Fixed Point Theorem, it follows that Φ has a fixed point in $K\bigcap(\bar{\Omega}_2\setminus\Omega_1)$.Further, since $G_1\left(t,s\right)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s\ge 0$, it follows that $u(t)>0$ for $0<t<1$.This completes the super-linear part of the theorem.

Theorem 3.2 Assume that $\left(H_1\right),\left(H_3\right)$ hold ,then the Problem $\left(1.1\right)$ and $\left(1.2\right)$ has at least one positive solution.

Proof Since (H_3) we first choose $r>0$ such that $f(t,u)\ge\mu u$, for $0\le u\le r$ where $\mu>0$ satisfies

$$\mu\sigma\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s\ge 1,$$

Let $\Omega_1=\{u\in C[0,1];\|u\|<r\}$, for $\forall u\in K\bigcap\partial\Omega_1$,from lemma 2.1, we have

$$\left\|\Phi u\left(t\right)\right\|\ge\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,s)G_2(s,\tau)f\left(\tau,u\left(\tau\right)\right)\mathrm{d}\tau\mathrm{d}s$$

$$\ge\sigma\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,s)G_2(s,\tau)f\left(\tau,u\left(\tau\right)\right)\mathrm{d}\tau\mathrm{d}s$$

$$\ge\mu\sigma\left\|u\right\|\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s\ge\left\|u\right\|$$

So that $\|\Phi u\|\ge\|u\|$.

Now since (H_3),there exists $R_1>0$ so that $f(t,u)\le\varepsilon u$, for $u\ge R_1$ where $\varepsilon>0$ satisfies

$\varepsilon\int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s<1$. $\forall u\in K\bigcap\partial\Omega_1$.

We consider two case:

Suppose $f(t,u)$ is unbounded for $\forall 0<u\le R$,we have $f(u)\le f(R)$, $R>\max\{r,\frac{R_1}{\sigma}\}$, $\min_{t\in[1/4,3/4]} u(t)\ge \sigma\|u\|=\sigma R>R_1$.

Let $\Omega_2=\{u\in C[0,1];\|u\|<R\}$, for $\forall u\in K\bigcap\partial\Omega_2$ therefore

$$\|\Phi u(t)\|\le\int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\mathrm{d}s$$

$$\le\int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)f(R)\mathrm{d}\tau\mathrm{d}s$$

$$\le\varepsilon R\int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)R\mathrm{d}\tau\mathrm{d}s\le R=\|u\|$$

So that $\|\Phi u\|\le\|u\|$.

Suppose $f(t,u)$ is bounded , there exists $N>0$, for $t\in[0,1]$ and $u\in[0,+\infty)$ we have $f(t,u)\le N$, $R>\max\{r,N\int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s\}$,

Let $\Omega_2=\{u\in C[0,1];\|u\|<R\}$,for $\forall u\in K\bigcap\partial\Omega_2$, from lemma 2.1 , we have

$$\|\Phi u(t)\|\le\int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\mathrm{d}s$$

$$\le N\int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s$$

$$\le R=\|u\|$$

So that $\|\Phi u\|\le\|u\|$.

Therefore, in either case we may put $\Omega_2=\{u\in C[0,1];\|u\|<R\}$ and for $\forall u\in K\bigcap\partial\Omega_2$ we have $\|\Phi u\|\le\|u\|$.By the second part of the Fixed Point Theorem it follows that BVP $(1.1),(1.2)$ has a positive solution, and this completes the proof of the theorem.

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