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Research Article

Existence and uniqueness of positive solutions for singular second-order Dirichlet boundary problems with impulse actions

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Abstract: By mixed monotone method, we establish the existence and uniqueness of positive solutions for singular second-order Dirichlet boundary problems with impulse actions. The theorems obtained are very general and complement previously known results.
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INTRODUCTION

Boundary value problems for ordinary differential equations are used to describe a large number of physical, biological and chemical phenomena. Many authors studied the existence and multiplicity of positive solutions for the boundary value problem of second-order differential equations(see [3]).In particular, the singular case was considered(see [2,4]).They mainly concerned with the existence and multiplicity of solutions using different methods. Recently, there were a few articles devoted to uniqueness problem by using the mixed monotone fixed point theorem(see [1,6]).However, they mainly investigated the case without impulse actions. Motivated by the work mentioned above, this paper attempts to study the existence and uniqueness of solutions for following singular boundary value problems with impulse actions. In this paper ,first we get a unique fixed point theorem for a class of mixed monotone operators. Our idea comes from the fixed point theorems for mixed monotone operators(see [5]). In virtue of the theorem, we consider the following singular second-order boundary problem with impulse effects:

$$\begin{cases} -x''=\lambda f(t,x), \quad t\neq t_k,\ 0<t<1, \lambda>0 \\ -\Delta x'|_{t=t_k}=I_k(x(t_k)), \quad k=1,2,\cdots,m, \\ \Delta x|_{t=t_k}=\bar{I}_k(x(t_k)), \quad k=1,2,\cdots,m, \\ x(0)=x(1)=0. \end{cases} \tag{1.1}$$

where $I=[0,1], 0<t_1<t_2<\cdots<t_m<1$ are given, $f\in C((0,1)\times(0,+\infty),(0,+\infty))$, $R^+=[0,+\infty), I_k, \bar{I}_k\in C(R^+,R^+), \Delta x'|_{t=t_k}=x'(t_k^+)-x'(t_k^-),\ \Delta x|_{t=t_k}=x(t_k^+)-x(t_k^-).x'(t_k^+),\ x(t_k^+)$ ($x'(t_k^-), x(t_k^-)$) denote the right limit (left limit) of $x'(t)$ *and* $x(t)$ at $t=t_k$ respectively, $f(t,x)$ and may be singular at $x=0$.

PRELIMINARY

Let P be a normal cone of a Banach space E ,and $e\in P$ with $\|e\|\leq 1, e\neq\theta$. Define

$$Q_e=\{x\in P\mid x\neq\theta, there\ \ exist\ \ constants\ \ m,M>0\ \ such\ \ that\ \ me\leq x\leq Me\}.$$

Now we give a definition(see [5]).

Definition 2.1. Assume $A:Q_e\times Q_e\to Q_e$. A is said to be mixed monotone if $A(x,y)$ is nondecreasing in x and nonincreasing in y, i.e. if $x_1\leq x_2(x_1,x_2\in Q_e)$ implies $A(x_1,y)\leq A(x_2,y)$ for any $y\in Q_e$, and

$y_1 \le y_2 (y_1, y_2 \in Q_e)$ implies $A(x, y_1) \ge A(x, y_2)$ for any $x \in Q_e$. $x^* \in Q_e$ is said to be a fixed point of A if $A(x^*, x^*) = x^*$.

Theorem 2.1(see [5]). Suppose that $A: Q_e \times Q_e \to Q_e$ is a mixed monotone operator and exists a constant $\alpha, 0 \le \alpha < 1,$ such that

$$A(tx, \frac{1}{t} y) \ge t^{\alpha} A(x, y) \qquad \forall x, y \in Q_e,\ 0 < t < 1. \tag{2.1}$$

Then A has a unique fixed point $x^* \in Q_e$. Moreover, for any $(x_0, y_0) \in Q_e \times Q_e$,

$$x_n = A(x_{n-1}, y_{n-1}),\ y_n = A(y_{n-1}, x_{n-1}), \qquad n = 1, 2 \cdots,$$

satisfy

$$x_n \to x^*, \quad y_n \to x^*$$

where

$$\| x_n - x^* \| = o(1 - r^{\alpha^n}), \quad \| y_n - y^* \| = o(1 - r^{\alpha^n}),$$

$0 < r < 1, r$ is a constant from (x_0, y_0).

Theorem 2.2(see [5]). Suppose that $A: Q_e \times Q_e \to Q_e$ is a mixed monotone operator and $\exists$ a constant $\alpha \in (0,1)$ such that (2.1) holds. If x_λ^* is a unique solution of equation

$$A(x, x) = \lambda x, \quad \lambda > 0,$$

in Q_e, then $\| x_\lambda^* - x_{\lambda_0}^* \| \to 0, \lambda \to \lambda_0$. If $0 < \alpha < \frac{1}{2}$, then $0 < \lambda_1 < \lambda_2$ implies $x_{\lambda_1}^* \ge x_{\lambda_2}^*$, $x_{\lambda_1}^* \ne x_{\lambda_2}^*$, and

$$\lim_{\lambda \to +\infty} \| x_\lambda^* \| = 0, \lim_{\lambda \to 0^+} \| x_\lambda^* \| = +\infty.$$

UNIQUENESS POSITIVE SOLUTION OF PROBLEM (1.1)

This section discusses the problem

$$\begin{cases} -x'' = \lambda f(t, x), \quad t \ne t_k,\ 0 < t < 1, \lambda > 0 \\ -\Delta x' |_{t=t_k} = I_k(x(t_k)), \qquad k = 1, 2, \cdots, m, \\ \Delta x |_{t=t_k} = \bar{I}_k(x(t_k)), \qquad k = 1, 2, \cdots, m, \\ x(0) = x(1) = 0. \end{cases}$$

Throughout this paper, we assume that

$$f(t, x) = q(t)[g(x) + h(x)], \quad t \in (0,1), \tag{3.1}$$

where

$$g: [0, +\infty) \to [0, +\infty) \ \textit{is continuous and nondecreasing};$$

$$h: (0, +\infty) \to (0, +\infty) \ \textit{is continuous and nonincreasing}; \tag{3.2}$$

and

$$I_k: [0, +\infty) \to [0, +\infty) \ \textit{is continuous and nondecreasing};$$

$$\bar{I}_k: [0, +\infty) \to [0, +\infty) \ \textit{is continuous and nondecreasing}. \tag{3.3}$$

Let $G(t, s)$ be the Green's function to $-x'' = 0,\ x(0) = x(1),$ we note that

$$G(t, s) := \begin{cases} (1-t)s, & 0 \le s \le t \le 1, \\ (1-s)t, & 0 \le t \le s \le 1. \end{cases}$$

And we can show that

$G(t, s) \le G(t, t) = t(1-t), G(t, s) \le G(s, s), G(t, s) \ge t(1-t) G(s, s),$ *for* $(t, s) \in [0,1] \times [0,1].$ Let $P = \{x \in C[0,1] \mid x(t) \ge 0,\ \forall t \in [0,1]\}.$ Obviously, P is a normal cone of Banach space $C[0,1]$.

Theorem 3.1. Suppose that there exists $\alpha \in (0,1)$ such that

$$g(tx) \geq t^{\alpha} g(x), \quad h(t^{-1}x) \geq t^{\alpha} h(x), \tag{3.4}$$

$$I_k(tx) \geq t^{\alpha} I_k(x), \quad \bar{I}_k(tx) \geq t^{\alpha} \bar{I}_k(x). \tag{3.5}$$

for any $t \in (0,1)$ and $x > 0$, and $q \in C((0,1),(0,\infty))$ satisfies

$$\int_0^1 s^{-\alpha}(1-s)^{-\alpha} q(s)ds < +\infty. \tag{3.6}$$

Then (1.1) has a unique positive solution $x_{\lambda}^{*}(t)$ And moreover, $0 < \lambda_1 < \lambda_2$ implies $x_{\lambda_1}^{*} \leq x_{\lambda_2}^{*}$, $x_{\lambda_1}^{*} \neq x_{\lambda_2}^{*}$,. If $\alpha \in (0, \frac{1}{2})$, then

$$\lim_{\lambda \to 0^{+}} \| x_{\lambda}^{*} \| = 0, \lim_{\lambda \to +\infty} \| x_{\lambda}^{*} \| = +\infty.$$

Proof Since (3.4) holds, let $t^{-1}x = y$, one has

$$h(y) \geq t^{\alpha} h(ty)$$

then

$$h(ty) \leq \frac{1}{t^{\alpha}} h(y), \ \forall\, t \in (0,1), \ y > 0. \tag{3.7}$$

Let $y = 1$. The above inequality is

$$h(t) \leq \frac{1}{t^{\alpha}} h(1), \ \forall\, t \in (0,1), \tag{3.8}$$

From (3.4),(3.7) and (3.8),one has

$$h(t^{-1}x) \geq t^{\alpha} h(x), \quad h(\frac{1}{t}) \geq t^{\alpha} h(1)$$

$$h(tx) \leq \frac{1}{t^{\alpha}} h(x), \quad h(t) \leq \frac{1}{t^{\alpha}} h(1), \ t \in (0,1), \ x > 0. \tag{3.9}$$

Similarly ,from (3.4), one has

$$g(tx) \geq t^{\alpha} g(x), \quad g(t) \geq t^{\alpha} g(1), t \in (0,1), \ x > 0. \tag{3.10}$$

Let $t = \frac{1}{x}$, $x > 1$. one has

$$g(x) \leq x^{\alpha} g(1), \quad x \geq 1. \tag{3.11}$$

Similarly, from (3.5),we also have

$$I_k(x) \leq x^{\alpha} I_k(1), \quad \bar{I}_k(x) \leq x^{\alpha} \bar{I}_k(1) \quad x \geq 1. \tag{3.12}$$

Let $e(t) = t(1-t)$, and we define

$$Q_e = \{x \in C[0,1] \mid \frac{1}{M} t(1-t) \leq x(t) \leq Mt(1-t), t \in [0,1]\}$$

where $M > 1$ is chosen such that

$$M > \max\{\{\lambda g(1)\int_0^1 q(s)ds + \lambda h(1)\int_0^1 s^{-\alpha}(1-s)^{-\alpha} q(s)ds + \sum_{k=1}^{m} (I_k(1) + \bar{I}_k(1))\}^{\frac{1}{1-\alpha}}$$

$$\{\lambda g(1)\int_0^1 G^{\alpha+1}(s,s)q(s)ds + \lambda h(1)\int_0^1 G(s,s)q(s)ds\}^{-\frac{1}{1-\alpha}}\}. \tag{3.13}$$

For any $x, y \in Q_e$, we define

$$A_{\lambda}(x,y)(t) = \lambda \int_0^1 G(t,s)q(s)[g(x(s)) + h(y(s))]ds + \sum_{0<t_k<t} G(t,t_k)(I_k(x(t_k)) + \bar{I}_k(x(t_k))), \quad \forall\, t \in [0,1]$$

(3.14)

First we show that $A_\lambda : Q_e \times Q_e \to Q_e$.

Let $x, y \in Q_e$, from (3.11) and (3.12)we have

$$g(x(t)) \le g(Mt(1-t)) \le g(M) \le M^\alpha g(1)$$

$$I_k(x(t_k)) \le I_k(Me(t_k)) \le I_k(M) \le M^\alpha I_k(1)$$

$$\overline{I}_k(x(t_k)) \le \overline{I}_k(Me(t_k)) \le \overline{I}_k(M) \le M^\alpha \overline{I}_k(1),\ t \in (0,1)$$

and from (3.9) we have

$$h(y(t)) \le h(\frac{1}{M}t(1-t)) \le t^{-\alpha}(1-t)^{-\alpha} h(\frac{1}{M})$$
$$\le M^\alpha t^{-\alpha}(1-t)^{-\alpha} h(1), \qquad t \in (0,1).$$

Then from (3.14) we have

$$A_\lambda(x,y)(t) \le t(1-t)\{\int_0^1 \lambda q(s)[M^\alpha g(1) + M^\alpha s^{-\alpha}(1-s)^{-\alpha} h(1)]ds + \sum_{k=1}^{m} M^\alpha (I_k(1) + \overline{I}_k(1))\}$$
$$= t(1-t)M^\alpha \{\lambda g(1)\int_0^1 q(s)ds + \lambda h(1)\int_0^1 s^{-\alpha}(1-s)^{-\alpha} q(s)ds + \sum_{k=1}^{m}(I_k(1) + \overline{I}_k(1))\}$$
$$\le t(1-t)M \qquad t \in [0,1]$$

On the other hand, for any $x, y \in Q_e$, from (3.9) and (3.10) ,we have

$$g(x(t)) \ge g(\frac{1}{M}t(1-t)) \ge t^\alpha (1-t)^\alpha g(\frac{1}{M}) \ge t^\alpha (1-t)^\alpha \frac{1}{M^\alpha} g(1),$$

and

$$h(y(t)) \ge h(Mt(1-t)) \ge h(M) = h(\frac{1}{\frac{1}{M}}) \ge \frac{1}{M^\alpha} h(1), \quad t \in (0,1)$$

Thus, from (3.14), we have

$$A_\lambda(x,y)(t) \ge t(1-t)\lambda\{\int_0^1 G(s,s)q(s)[g(x(s)) + h(y(s))]ds$$
$$+ \sum_{0<t_k<t} G(t_k,t_k)(I_k(x(t_k)) + \overline{I}_k(x(t_k)))\}$$
$$\ge t(1-t)\lambda\int_0^1 G(s,s)q(s)[s^\alpha(1-s)^\alpha \frac{1}{M^\alpha} g(1) + \frac{1}{M^\alpha} h(1)]ds$$
$$= t(1-t)\frac{1}{M^\alpha}[\lambda g(1)\int_0^1 G(s,s)q(s)s^\alpha(1-s)^\alpha ds + \lambda h(1)\int_0^1 G(s,s)q(s)ds]$$
$$\ge t(1-t)\frac{1}{M}$$

So, A_λ is well defined and $A_\lambda(Q_e \times Q_e) \subset Q_e$

Next, for any $l \in (0,1)$, one has

$$A_\lambda(lx, l^{-1}y)(t) = \lambda\int_0^1 G(t,s)q(s)[g(lx(s)) + h(l^{-1}y(s))]ds + \sum_{0<t_k<t} G(t,t_k)(I_k(lx(t_k)) + \overline{I}_k(lx(t_k)))$$
$$\ge \lambda\int_0^1 G(t,s)q(s)[l^\alpha g(x(s)) + l^\alpha h(y(s))]ds + \sum_{0<t_k<t} l^\alpha G(t,t_k)(I_k(x(t_k)) + \overline{I}_k(x(t_k)))$$
$$= l^\alpha A_\lambda(x,y)(t), \qquad t \in [0,1]$$

So the conditions of Theorems 2.1 and 2.2 hold. Therefore there exists a unique $x_\lambda^* \in Q_e$ such that $A_\lambda(x^*, x^*) = x_\lambda^*$. It is easy to check that x_λ^* is a unique positive solution of (1.1) for given $\lambda > 0$ .Moreover, Theorem 2.2 means that if $0 < \lambda_1 < \lambda_2$ then $x_{\lambda_1}^*(t) \le x_{\lambda_2}^*(t),\ x_{\lambda_1}^*(t) \ne x_{\lambda_2}^*(t)$, and if $\alpha \in (0, \frac{1}{2})$,then

$$\lim_{\lambda \to 0^+} \| x_\lambda^* \| = 0, \lim_{\lambda \to +\infty} \| x_\lambda^* \| = +\infty.$$

This completes the proof.

Example. Consider the following singular second-order boundary value problem:

$$\begin{cases} -x'' = \lambda(\mu x^a + x^{-b}), & t \ne t_k,\ 0 < t < 1, \\ -\Delta x'|_{t=t_k} = a_k x^c(t_k), \quad a_k \ge 0, & k = 1, 2, \cdots, m, \\ \Delta x|_{t=t_k} = b_k x^d(t_k), \quad b_k \ge 0, & k = 1, 2, \cdots, m, \\ x(0) = x(1) = 0, \end{cases} \tag{3.15}$$

where $\lambda, a, b, c, d > 0,\ \mu \ge 0,\ \max\{a, b, c, d\} < 1$, Let

$$\alpha = \max\{a, b, c, d\}, q(t) = 1, g(x) = \mu x^a, h(x) = x^{-b}, I_k(x) = a_k x^c, \overline{I}_k(x) = b_k x^d.$$

Thus $0 < \alpha < 1$ and

$$g(tx) = t^a g(x) \ge t^\alpha g(x), \quad h(t^{-1}x) = t^b h(x) \ge t^\alpha h(x)$$

$$I_k(tx) = t^c I_k(x) \ge t^\alpha I_k(x), \quad \overline{I}_k(tx) = t^d I_k(x) \ge t^\alpha \overline{I}_k(x).$$

for any $t \in (0,1)$ $x > 0$, and $\int_0^1 s^{-\alpha}(1-s)^{-\alpha} ds < +\infty$,

Thus all conditions in Theorem 3.1 are satisfied. In addition, $0 < \lambda_1 < \lambda_2$ implies $x_{\lambda_1}^* \le x_{\lambda_2}^*$,, $x_{\lambda_1}^* \ne x_{\lambda_2}^*$. If $\max\{a, b, c, d\} \in (0, \frac{1}{2})$, then

$$\lim_{\lambda \to 0^+} \| x_\lambda^* \| = 0, \lim_{\lambda \to +\infty} \| x_\lambda^* \| = +\infty.$$

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