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Research Article

The calculation method for a kind of definite integral

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Abstract: This paper discussed $a_n = \frac{2}{\pi}\int_{-1}^{1}\frac{\arctan x \cdot T_n(x)}{\sqrt{1-x^2}}dx$ this kind of definite integral. ($T_n(x)$ is n-th Qibixiaofu polynomial, $n = 0,1,2\ldots$) The conclusion is when n is even number, $a_n = a_{2k} = 0$. And when n is odd number,

$$a_n = a_{2k+1} = (-1)^k \frac{2}{2k+1}(\sqrt{2}-1)^{2k+1}(k = 0,1,2\ldots)$$

Keywords: Definite integral；odd function, Mathematical induction；Qibixiaofu series, error estimation.

1. When n is even number, the numerical value of

$$a_n = \frac{2}{\pi}\int_{-1}^{1}\frac{\arctan x \cdot T_n(x)}{\sqrt{1-x^2}}dx$$

When n is even number, $T_n(x)$ is even function, and $\arctan x$ is odd function, $\sqrt{1-x^2}$ is even function, so the integrand $\frac{\arctan x \cdot T_n(x)}{\sqrt{1-x^2}}$ of this integral is odd function. As the integrating range [-1, 1] is symmetric, so

$$a_n = a_{2k} = 0 \quad (k = 0,\ 1,\ 2,\ \ldots) \tag{1}$$

2. When n is odd number, the numerical value of

$$a_n = \frac{2}{\pi}\int_{-1}^{1}\frac{\arctan x \cdot T_n(x)}{\sqrt{1-x^2}}dx$$

1) The relationship between the definite integral a_{2k+1}, $a_{2(k-1)+1}$ and $a_{2(k-2)+1}$.

If n is odd number, $x = \cos t$, then

$$\begin{aligned}
a_{2k+1} &= \frac{2}{\pi}\int_{-1}^{1}\frac{\arctan x \cdot T_{2k+1}(x)}{\sqrt{1-x^2}}dx \\
&= \frac{2}{\pi}\int_{0}^{\pi}\arctan(\cos t)\cos(2k+1)t\,dt \\
&= \frac{2}{\pi}\left[\frac{1}{2k+1}\arctan(\cos t)\sin(2k+1)t\right]_0^{\pi} + \frac{2}{(2k+1)\pi}\int_0^{\pi}\frac{\sin(2k+1)t\sin t}{1+\cos^2 t}dt \\
&= \frac{2}{(2k+1)\pi}\int_0^{\pi}\frac{\sin(2k+1)t\sin t}{1+\cos^2 t}dt
\end{aligned} \tag{2}$$

So we can work out the follow equation.

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \frac{\sin^2 t}{1+\cos^2 t} dt$$

$$= \frac{2}{\pi}\lim_{\varepsilon \to +0}\left[\int_0^{\frac{\pi}{2}-\varepsilon} \frac{\sin^2 t}{1+\cos^2 t} dt + \int_{\frac{\pi}{2}+\varepsilon}^{\pi} \frac{\sin^2 t}{1+\cos^2 t} dt \right]$$

$$= \frac{2}{\pi}\lim_{\varepsilon \to +0}\left\{ \left[\sqrt{2}\arctan(\frac{\tan t}{\sqrt{2}}) - t\right]_0^{\frac{\pi}{2}-\varepsilon} + \left[\sqrt{2}\arctan(\frac{\tan t}{\sqrt{2}}) - t\right]_{\frac{\pi}{2}+\varepsilon}^{\pi} \right\}$$

$$= \frac{2}{\pi}\left[(\frac{\sqrt{2}}{2}\pi - \frac{\pi}{2}) + (-\pi + \frac{\sqrt{2}}{2}\pi + \frac{\pi}{2}) \right] = 2(\sqrt{2}-1) \quad (3)$$

(When $t = \frac{\pi}{2}$, $\tan t$ has no definition)

$$a_3 = \frac{2}{3\pi}\int_0^{\pi} \frac{\sin 3t \sin t}{1+\cos^2 t} dt = \frac{2}{3\pi}\int_0^{\pi} \frac{(-4\sin^3 t + 3\sin t)\sin t}{1+\cos^2 t} dt$$

$$= \frac{2}{3\pi}\left[-4\int_0^{\pi} \frac{\sin^4 t}{1+\cos^2 t} dt + 3\int_0^{\pi} \frac{\sin^2 t}{1+\cos^2 t} dt\right]$$

Because

$$\int_0^{\pi} \frac{\sin^4 t}{1+\cos^2 t} dt = \int_0^{\pi} \frac{\sin^2\left[2-(1+\cos^2 t)\right]}{1+\cos^2 t} dt = 2\int_0^{\pi} \frac{\sin^2 t}{1+\cos^2 t} dt - \int_0^{\pi} \sin^2 t dt$$

Though (3), we can figure out $\int_0^{\pi} \frac{\sin^2 t}{1+\cos^2 t} dt = \pi(\sqrt{2}-1)$, and as

$$\int_0^{\pi} \sin^2 t dt = \frac{\pi}{2},$$

so

$$a_3 = \frac{2}{3\pi}\left\{-4\left[2(\sqrt{2}-1)\pi - \frac{\pi}{2}\right] + 3(\sqrt{2}-1)\pi\right\} = -\frac{1}{3}(10\sqrt{2}-14) \quad (4)$$

From the two trigonometric functions as follows,

$$\sin(2k+1)t = \sin\left[2(k-1)+1\right]t\cos 2t + \cos\left[2(k-1)+1\right]t\sin 2t$$

$$\sin\left[2(k-2)+1\right]t = \sin\left[2(k-1)+1\right]t\cos 2t - \cos\left[2(k-1)+1\right]t\sin 2t$$

We can figure out

$$\sin(2k+1)t = 2\sin\left[2(k-1)+1\right]t\cos 2t - \sin\left[2(k-2)+1\right]t$$

Multiply the two sides of this equation by $\frac{2\sin t}{(2k+1)\pi(1+\cos^2 t)}$, and calculate by integration.

$$\frac{2}{(2k+1)\pi}\int_0^{\pi} \frac{\sin(2k+1)t\sin t}{1+\cos^2 t} dt$$

$$= \frac{2(k-1)+1}{2k+1}\cdot\frac{2}{\left[2(k-1)+1\right]\pi}\cdot 2\int_0^{\pi} \frac{\sin\left[2(k-1)+1\right]\sin t\cos 2t}{1+\cos^2 t} dt$$

$$-\frac{2(k-2)+1}{2k+1}\cdot\frac{2}{\left[2(k-2)+1\right]\pi}\int_0^{\pi} \frac{\sin\left[2(k-2)+1\right]t\sin t}{1+\cos^2 t} dt$$

So

$$a_{2k+1}=-\frac{2k-1}{2k+1}\cdot\frac{2}{\left[2(k-1)+1\right]\pi}\cdot 2\int_0^{\pi}\frac{\sin\left[2(k-1)+1\right]t\sin t(1-2\sin^2 t)}{1+\cos^2 t}dt-\frac{2k-3}{2k+1}a_{2(k-2)+1} \quad (5)$$

Since

$$\int_0^{\pi}\frac{\sin[2(k-1)+1]t\sin t(1-2\sin^2 t)}{1+\cos^2 t}dt$$

$$=\int_0^{\pi}\frac{\sin[2(k-1)+1]t\sin t}{1+\cos^2 t}dt-2\int_0^{\pi}\frac{\sin[2(k-1)+1]t\sin t\left[2-(1+\cos^2 t)\right]}{1+\cos^2 t}dt$$

$$=-3\int_0^{\pi}\frac{\sin[2(k-1)+1]t\sin t}{1+\cos^2 t}dt+2\int_0^{\pi}\sin[2(k-1)+1]t\sin t dt$$

From(5)we can figure out

$$a_{2k+1}=-6\frac{2k-1}{2k+1}a_{2(k-1)+1}-\frac{2k-3}{2k+1}a_{2(k-2)+1}\ \ (k=2,3,4\Lambda) \quad (6)$$

2) When n is odd number，The calculation formula of a_n

Now we can prove the follow equation by induction

$$a_{2k+1}=(-1)^k\frac{2(\sqrt{2}-1)^{2k+1}}{2k+1}\quad (k=0,1,2\Lambda) \quad (7)$$

From (3) We can know that When $k=0$,

$$a_1=2(\sqrt{2}-1)=(-1)^0\frac{2(\sqrt{2}-1)^{2\cdot 0+1}}{2\cdot 0+1}$$

From (4) We can know that When $k=1$,

$$a_3=-\frac{1}{3}(10\sqrt{2}-14)=(-1)^1\frac{2(\sqrt{2}-1)^{2\cdot 1+1}}{2\cdot 1+1}$$

So when $k=0,1$，the equation (7) may be tenable.

If $k=m-1$，$k=m$,（7）is tenable，so

$$a_{2m+1}=(-1)^m\frac{2(\sqrt{2}-1)^{2m+1}}{2m+1}\ ;\ a_{2(m-1)+1}=(-1)^{m-1}\frac{2(\sqrt{2}-1)^{2(m-1)+1}}{2(m-1)+1}\ \text{is tenable.} \quad (8)$$

Form (6) we can work out that, when $k=m+1$,

$$a_{2(m+1)+1}=-6\frac{2m+1}{2(m+1)+1}a_{2m+1}-\frac{2(m-1)+1}{2(m+1)+1}a_{2(m-1)+1}$$

Form（8）we can figure out by induction hypothesis,

$$a_{2(m+1)+1}=-6\frac{2m+1}{2(m+1)+1}\cdot\frac{(-1)^m 2(\sqrt{2}-1)^{2m+1}}{2m+1}-\frac{2(m-1)+1}{2(m+1)+1}\cdot\frac{(-1)^{m-1}2(\sqrt{2}-1)^{2(m-1)+1}}{2(m-1)+1}$$

$$=6(-1)^{m+1}\frac{2(\sqrt{2}-1)^{2m+1}}{2(m+1)+1}+\frac{(-1)^m 2(\sqrt{2}-1)^{2(m-1)+1}}{2(m+1)+1}=(-1)^m\frac{2(\sqrt{2}-1)^{2m-1}}{2(m+1)+1}\left[-6(\sqrt{2}-1)^2+1\right]$$

$$=(-1)^m\frac{2(\sqrt{2}-1)^{2m-1}}{2(m+1)+1}\left[-(\sqrt{2}-1)^4\right]$$

$$=(-1)^{m+1}\frac{2(\sqrt{2}-1)^{2(m+1)+1}}{2(m+1)+1},$$

So the equation (7) can be tenable.

From（1） and（6）, we can figure out

$$a_n = \frac{2}{\pi}\int_{-1}^{1}\frac{\arctan x \cdot T_n(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0, & n = 2k \\ (-1)^k \dfrac{2(\sqrt{2}-1)^{2k+1}}{2k+1}, & n = 2k+1 \end{cases} (k = 0,1,2\Lambda) \qquad (9)$$

3. Application

By the definition of Qibixiaofu function series, the Qibixiaofu series of $\arctan x$ $(-1 \le x \le 1)$ is

$$\frac{a_0}{2}T_0(x) + a_1T_1(x) + a_2T_2(x) + \Lambda$$

Among them, $$a_n = \frac{2}{\pi}\int_{-1}^{1}\frac{\arctan x\, T_n(x)}{\sqrt{1-x^2}}dx \quad (n = 0,1,2\Lambda)$$

From（9）we can figure our the value of a_n $(n = 0,1,2\Lambda)$ is

$$a_0 = a_2 = a_4 = \Lambda = a_{2k} = \Lambda = 0\ ;$$

$$a_1 = 2(\sqrt{2}-1),\ a_3 = -\frac{2}{3}(\sqrt{2}-1)^3,\ a_3 = \frac{2}{5}(\sqrt{2}-1)^5, \ldots,\ a_{2k+1} = \frac{2(-1)^k}{2k+1}(\sqrt{2}-1)^{2k+1}, \Lambda\ .$$

So the Qibixiaofu series of $arctgx$ is

$$2(\sqrt{2}-1)T_1(x) - \frac{2}{3}(\sqrt{2}-1)^3T_3(x) + \frac{2}{5}(\sqrt{2}-1)^5T_5(x) + \mathrm{L} + (-1)^k\frac{2(\sqrt{2}-1)^{2k+1}}{2k+1}T_{2K+1}(x) + \mathrm{L}$$

1.1.1. And the Qibixiaofu series of $\arctan x$ partial sum is

$$P_{2k+1}(x) = 2(\sqrt{2}-1)T_1(x) - \frac{2}{3}(\sqrt{2}-1)^3T_3(x) + \frac{2}{5}(\sqrt{2}-1)^5T_5(x) + \mathrm{L} + (-1)^k\frac{2(\sqrt{2}-1)^{2k+1}}{2k+1}T_{2K+1}(x)$$

It is the best mean square approximation polynomial of $\arctan x$. Taking $P_{2k+1}(x)$ as an approximation of $\arctan x$.

$$\arctan x \approx 2(\sqrt{2}-1)\left[T_1(x) - \frac{1}{3}(\sqrt{2}-1)^2T_3(x) + \mathrm{L} + (-1)^k\frac{2(\sqrt{2}-1)^{2k}}{2k+1}T_{2K+1}(x)\right] \qquad (10)$$

And taking $P_5(x)$ as an approximation of $\arctan x$,calculate the approximation of $\arctan 0.5$.

As

$$T_1(x) = x,\ T_3(x) = 4x^3 - 3x,\ T_5(x) = 16x^5 - 20x^3 + 5x,$$

so

$$\arctan 0.5 \approx 2(\sqrt{2}-1)\times 0.5 + \frac{1}{3}(14-10\sqrt{2})(4\times 0.5^3 - 3\times 0.5) +$$

$$\frac{1}{5}(58\sqrt{2}-82)(16\times 0.5^5 - 20\times 0.5^3 + 5\times 0.5)$$

$$= (70\sqrt{2}-98)\times 0.5 + \frac{1}{3}(1040-736\sqrt{2})\times 0.5^3 + \frac{16}{5}(58\sqrt{2}-82)\times 0.5^5$$

$$\approx 0.994949367\times 0.5 - 0.287060633\times 0.5^3 + 0.078037184\times 0.5^5$$

$$= 0.462198959$$

Now try to make $P_{2k+1}(x)$ which is the best mean square approximation polynomial of $\arctan x$ instead of $\arctan x$ error estimation. Since $\arctan x$ is a continuous function， and its first-order derivative $(\arctan x)'$ is bounded, so the Qibixiaofu series of $\arctan x$ uniform convergence in $\arctan x$.

$$\arctan x = a_1T_1(x) + a_3T_3(x) + \mathrm{L} + a_{2k+1}T_{2k+1}(x) + \mathrm{L} \qquad (11)$$

By (11),

$$\left|\arctan x - P_{2k+1}(x)\right| = \left|a_{2(k+1)+1}T_{2(k+1)+1}(x) + a_{2(k+2)+1}T_{2(k+2)+1}(x) + \mathrm{L}\ \right|$$

$$\leq \left| a_{2(k+1)+1} T_{2(k+1)+1}(x) \right| + \left| a_{2(k+2)+1} T_{2(k+2)+1}(x) \right| + \Lambda$$

As

$$|T_n(x)| \leq 1 \quad (n = 0,1,2,\Lambda),$$

So

$$\left|\arctan x - P_{2k+1}(x)\right| \leq \left|a_{2(k+1)+1}\right| + \left|a_{2(k+2)+1}\right| + \mathrm{L} \qquad (12)$$

By (7), we can figure out

$$|a_1| + |a_3| + \Lambda + |a_{2k+1}| + \Lambda = 2\left[(\sqrt{2}-1) + \frac{1}{3}(\sqrt{2}-1)^3 + \Lambda + \frac{(\sqrt{2}-1)^{2k+1}}{2k+1} + \Lambda \right]$$

By the formula

$$\ln\frac{1+x}{1-x} = 2\left(x + \frac{x^2}{3} + \frac{x^5}{5} + \Lambda + \frac{x^{2k+1}}{2k+1} + \Lambda\right) \quad (-1 < x < 1)$$

Can work out

$$|a_1| + |a_3| + \Lambda + |a_{2k+1}| + \Lambda = \ln\frac{1+(\sqrt{2}-1)}{1-(\sqrt{2}-1)} = \ln(\sqrt{2}+1) \qquad (13)$$

Though（12） and（13）, we can know

$$\left|\arctan x - P_{2k+1}(x)\right| \leq \ln(\sqrt{2}+1) - \left[|a_1| + |a_3| + \mathrm{L}\ |a_{2k+1}|\right]$$

$$= \ln(\sqrt{2}+1) - 2(\sqrt{2}-1)\left[1 + \frac{1}{3}(\sqrt{2}-1)^2 + \frac{1}{5}(\sqrt{2}-1)^4 + \Lambda + \frac{1}{2k+1}(\sqrt{2}-1)^{2k}\right]$$

$$(k = 0,1,2\Lambda) \qquad (14)$$

Now we use $P_5(x)$, $P_7(x)$、$P_9(x)$ instead of $\arctan x$ error estimation in [-1, 1].Though (14), we can figure out

$$\left|\arctan x - P_5(x)\right| \leq \ln(\sqrt{2}+1) - 2(\sqrt{2}-1)\left[1 + \frac{1}{3}(\sqrt{2}-1)^2 + \frac{1}{5}(\sqrt{2}-1)^4\right] \leq 7\times 10^{-4}$$

$$\left|\arctan x - P_7(x)\right| \leq \ln(\sqrt{2}+1) - 2(\sqrt{2}-1)\left[1 + \frac{1}{3}(\sqrt{2}-1)^2 + \frac{1}{5}(\sqrt{2}-1)^4 + \frac{1}{7}(\sqrt{2}-1)^6\right] \leq 10^{-4}$$

$$\left|\arctan x - P_9(x)\right| \leq \ln(\sqrt{2}+1) - 2(\sqrt{2}-1)\left[1 + \frac{1}{3}(\sqrt{2}-1)^2 + \frac{1}{5}(\sqrt{2}-1)^4 + \frac{1}{7}(\sqrt{2}-1)^6 + \frac{1}{9}(\sqrt{2}-1)^8\right] \leq 14\times 10^{-6}$$

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