Research Article

Existence of Eigenvalue to Boundary Value Problems for Nonlinear High-Order Differential Equations

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Abstract: In this paper, investigation of positive solutions for the boundary value problem of eigenvalue problem has been reported. It is studied by employing the positive property of the Green's function, the fixed point theorem and Krasnoselskii fixed point theorem in cone.

Keywords: Higher order boundary value problem, Positive solution, Cone fixed point theorem.

INTRODUCTION

Boundary value problem has been an important branch in the theory of differential equations, and it is a very active research field at present. The research on the eigenvalue problem has attracted many scholars. The features of the high order nonlinear boundary value problems of the existence of value enrich the problem [2-6]. However, most authors study the problem under the conjugate boundary conditions or more simple boundary conditions, and papers constructed the research on high-order ordinary differential equations with more complex boundary conditions are not common. This paper is different from theirs, and study the eigenvalue for nonlinear high-order differential equations has been done. It is proved that the equation has at least one positive solution through the cone fixed point theorem.

Preliminary Notes

In this paper, we concern on the following nonlinear higher-order boundary value problem

$$\begin{cases} (-1)^m y^{(2m)} = \lambda f(x, y(x)), \\ y^{(i)}(0) = 0, 0 \le i \le m-1, \\ y^{(j)}(1) = 0, m \le j \le 2m-1. \end{cases} \tag{1}$$

where $\lambda > 0$, let

(H_1) $f(x, y)$ is not equal to zero any where for any compact subinterval in $[0,1]$, and $\int_0^1 x^{m-1} f(x, y)\mathrm{d}x < +\infty$;

(H_2) $\lim\limits_{y \to 0^+} \dfrac{f(x, y)}{y} = f_0, \lim\limits_{y \to +\infty} \dfrac{f(x, y)}{y} = f_\infty$.

Theorem 2.1 Let B be a Banach space and K is a cone in B. Assume that Ω_1 and Ω_2 are open subsets of B with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$ .Let $\Phi : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ be a completely continuous operator such that either

(i) $\|\Phi y\| \le \|y\|, y \in K \cap \partial\Omega_1, \|\Phi y\| \ge \|y\|, y \in K \cap \partial\Omega_2$; or

(ii) $\|\Phi y\| \ge \|y\|, y \in K \cap \partial\Omega_1, \|\Phi y\| \le \|y\|, y \in K \cap \partial\Omega_2$.

Then, Φ has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Let $A = (\alpha \int_{\frac{1}{4}}^{\frac{3}{4}} g(s)\mathrm{d}s)^{-1}$, $B = (\| \beta \| \int_0^1 g(s)\mathrm{d}s)^{-1}$.

3 Main Results

To obtain positive solutions for the problem (1), we state some properties of Green's function for (1).

We can easily obtain it, the problem (1)is equivalent to the integral equation

$$y(x) = \lambda \int_0^1 G(x,s) f(s, y(s)) \mathrm{d}s \tag{2}$$

where

$$G(x,s) = \begin{cases} \dfrac{1}{[(m-1)!]^2} \displaystyle\int_0^s u^{m-1}(u+x-s)^{m-1} \mathrm{d}u, 0 \le s \le x \le 1 \\[3mm] \dfrac{1}{[(m-1)!]^2} \displaystyle\int_0^x u^{m-1}(u+s-x)^{m-1} \mathrm{d}u, 0 \le x \le s \le 1 \end{cases} \tag{3}$$

Moreover, the following results have been recently offered by [3].

Lemma 2.1 For any $x, s \in [0,1]$, we have

$$\alpha(x)\, g(s) \le G(x,s) \le \beta(x) g(s) ,$$

Where

$$\alpha(x) = \frac{x^m}{2m-1} \; ; \; \beta(x) = \frac{x^{m-1}}{m} \; ; \; g(s) = \frac{1}{[(m-1)!]^2} s^m .$$

Define the cone K in Banach space $C[0,1]$, given by

$$K = \{ y \in C[0,1] : y(x) \ge 0, \min_{\frac{1}{4} \le x \le \frac{3}{4}} y(x) \ge \alpha \|y\| / \|\beta\| \} ,$$

where $\|y\| = \sup_{0 \le y \le 1} | y(x)|,$ $\alpha = \min_{\frac{1}{4} \le x \le \frac{3}{4}} \alpha(x),$ $\| \beta \| = \max_{0 \le x \le 1} \beta(x).$

We define the operator $\Phi : K \to K$ by

$$(\Phi y)(x) = \lambda \int_0^1 G(x,s) f(s, y(s)) \mathrm{d}s .$$

For any $y \in K$, we have

$$\min_{\frac{1}{4} \le x \le \frac{3}{4}} \Phi y(x) \ge \lambda \min_{\frac{1}{4} \le x \le \frac{3}{4}} \int_0^1 \alpha(x) g(s) f(s, y(s)) \mathrm{d}s$$

$$\ge \frac{\alpha}{\|\beta\|} \max_{0 \le x \le 1} \int_0^1 \lambda G(x,s) f(s, y(s)) \mathrm{d}s = \frac{\alpha}{\|\beta\|} \|\Phi y\|$$

This implies $\Phi(K) \subset K$.

We can easily obtain it, $\Phi : K \to K$ is a completely continuous mapping.

Theorem 2.2 Assume that (H_1) and (H_2) hold, Our assumption throughout is,

$$\frac{A}{f_\infty} < \lambda < \frac{B}{f_0}$$

then the problem(1) has at least one positive solution.

Proof Choose $\varepsilon > 0$, such that

$$[\alpha(f_\infty - \varepsilon)\int_{\frac{1}{4}}^{\frac{3}{4}} g(s) \mathrm{d}s]^{-1} \le \lambda \le [\| \beta \| (f_0 + \varepsilon)\int_0^1 g(s) \mathrm{d}s]^{-1}$$

Following from $\lim_{y \to 0^+} \dfrac{f(x, y)}{y} = f_0$, there exists $\delta_1 > 0$, such that

$$f(x, y) \le (f_0 + \varepsilon) y, 0 < y < \delta_1 .$$

Let $\Omega_1 = \{ y \in [0,1], \|y\| < \delta_1 \}$, for all $x \in [0,1]$, for any $y \in K \cap \partial\Omega_1$,

$$\Phi y(x) \le \lambda \|\beta\| \int_0^1 g(s)(f_0 + \varepsilon) \|y\| \mathrm{d}s \le \|y\|$$

Thus $\| \Phi y \| \le \|y\|$.

Following from $\lim\limits_{y\to+\infty}\dfrac{f(x,y)}{y}=+\infty$, there exists $N>0$, such that $f(x,y)\geq(f_\infty-\varepsilon)y$, $y\geq N$

Choose $R>max\{\delta_1,\dfrac{\|\beta\|}{\alpha}N\}$, let $\Omega_2=\{y\in\mathrm{C}[0,1],\|y\|<R\}$,

For all $x\in[0,1]$, $\min\limits_{\frac{1}{4}\leq x\leq\frac{3}{4}}y(x)\geq\dfrac{\alpha}{\|\beta\|}\|y\|=\dfrac{\alpha}{\|\beta\|}R\geq N$, $y\in K\cap\partial\Omega_2$

$$\Phi y(x)\geq\lambda\int_{\frac{1}{4}}^{\frac{3}{4}}\alpha\mathrm{g}(s)f(s,y(s))\mathrm{d}s\geq\lambda\alpha\int_{\frac{1}{4}}^{\frac{3}{4}}\mathrm{g}(s)\mathrm{d}s\cdot(f_\infty-\varepsilon)\|y\|\geq\|y\|$$

Thus $\|\Phi y\|\geq\|y\|$.

Thus, theorem 2.1 implies, the problem(1) has at least one positive solution.

Theorem 2.3 Assume that (H_1) and (H_2) hold. Our assumption throughout is, $\dfrac{A}{f_0}<\lambda<\dfrac{B}{f_\infty}$

then the problem(1) has at least one positive solution.

Proof Choose $\varepsilon>0$, such that

$$[\alpha\int_{\frac{3}{4}}^{\frac{1}{4}}g(s)\mathrm{d}s(f_0-\varepsilon)]^{-1}\leq[\|\beta\|\int_0^1 g(s)\mathrm{d}s(f_\infty+\varepsilon)]^{-1}$$

Following from $\lim\limits_{y\to 0^+}\dfrac{f(x,y)}{y}=f_0$, there exists $\delta_1>0$, such that

$$f(x,y)\geq(f_0-\varepsilon)y,0<y<\partial_1.$$

Let $\Omega_1=\{y\in[0,1],\|y\|<\delta_1\}$, for all $x\in[0,1]$, for any $y\in K\cap\partial\Omega_1$,

$$\Phi y(x)\geq\lambda\int_{\frac{1}{4}}^{\frac{3}{4}}G(x,s)f(s,y)\mathrm{d}s\geq\lambda\alpha\int_{\frac{1}{4}}^{\frac{3}{4}}g(s)\mathrm{d}s\cdot(f_0-\varepsilon)\|y\|\geq\|y\|$$

Thus $\|\Phi y\|\geq\|y\|$.

Following from $\lim\limits_{y\to+\infty}\dfrac{f(x,y)}{y}=+\infty$, there exists $N>0$, for any $y\geq N$, such that $f(x,y)\leq(f_\infty+\varepsilon)y$.

We consider two cases：f is bounded or f is unbounded.

Situation(1): If f is bounded, there exists $M>0$，such that
$$f(x,y)\leq M,\quad 0<y<\infty,$$

Choose $R_2=\max\{2R_1,M\|\beta\|\lambda\int_0^1 g(s)\mathrm{d}s\}$, if for all $y\in K$ and $\|y\|=R_2$,

$$\Phi y(x)\leq\lambda\|\beta\|\int_0^1 g(s)f(s,y)\mathrm{d}s\leq\lambda\|\beta\|M\int_0^1 g(s)\mathrm{d}s\leq R_2=\|y\|.$$

Let $\Omega_2=\{y\in C[0,1],\|y\|<R_2$, for all $x\in[0,1]$, $\|\Phi y\|\leq\|y\|$, $y\in K\cap\partial\Omega_2$.

Situation(2): If f is unbounded，choose $R_2>\max\{2R_1,N\}$，such that
$$f(x,y)\leq\sup\limits_{0\leq x\leq 1}f(x,R_2),0\leq x\leq 1,0\leq y\leq R_2$$

if for all $y\in K$ and $\|y\|=R_2$, we get

$$\|\Phi y\|\leq\lambda\|\beta\|\int_0^1 g(s)f(s,R_2)\mathrm{d}s\leq\lambda\|\beta\|\int_0^1 g(s)\mathrm{d}s\cdot(f_\infty+\varepsilon)\|y\|\leq\|y\|.$$

Let $\Omega_2=\{y\in C[0,1],\|y\|<R_2$, for all $x\in[0,1]$, $\|\Phi y\|\leq\|y\|$, $y\in K\cap\partial\Omega_2$.

From the above two kinds of circumstances，Choose $\Omega_2=\{y\in C[0,1],\|y\|<R_2$, for any $x\in[0,1]$, such that

$$\|\Phi y\|\leq\|y\|,\ y\in K\cap\partial\Omega_2.$$

Thus, theorem 2.1 implies, the problem(1) has at least one positive solution.

Lemma 2.2 Assume that (H_1) and (H_2) hold. Let $f_\infty = +\infty$ .If λ is sufficiently large, the problem(1) has no positive solution.

Proof Following from $\lim\limits_{y \to +\infty} \dfrac{f(x,y)}{y} = +\infty$, there exists $M > 0$, such that

$$f(x,y) > Ry(x), \quad y \geq M .$$

If $\| y \| \geq \dfrac{M}{\alpha} \| \beta \|$, since $\min\limits_{x \in [\frac{1}{4}, \frac{3}{4}]} y(x) \geq \dfrac{\alpha}{\| \beta \|} \| y \| \geq M$, we get

$$\| y \| \geq \lambda \min_{x \in [\frac{1}{4}, \frac{3}{4}]} \alpha(x) \int_{\frac{1}{4}}^{\frac{3}{4}} g(s) f(s, y(s)) \mathrm{d}s$$

$$\geq \lambda \min_{x \in [\frac{1}{4}, \frac{3}{4}]} \alpha(x) \int_{\frac{1}{4}}^{\frac{3}{4}} g(s) R y(s) \mathrm{d}s$$

$$\geq \lambda R \frac{\alpha^2}{\| \beta \|} \| y \| \int_{\frac{1}{4}}^{\frac{3}{4}} g(s) \mathrm{d}s$$

Thus, $\lambda \leq (R \dfrac{\alpha^2}{\| \beta \|} \int_{\frac{1}{4}}^{\frac{3}{4}} g(s)\mathrm{d}s)^{-1}$, which contradicts with λ sufficiently large.

If $\| y \| < \dfrac{M}{\alpha} \| \beta \|$, let $c = \int_0^1 g(s) f(s, y(s))\mathrm{d}s$, then $0 < c < +\infty$, and

$\dfrac{M}{\alpha} \| \beta \| > \| y \| > \lambda \min\limits_{x \in [\frac{1}{4}, \frac{3}{4}]} \alpha(x) \int_0^1 g(s) f(s, y(s))\mathrm{d}s \geq \lambda \alpha c$, i.e. $\lambda \leq M \| \beta \| (\alpha^2 c)^{-1}$, which is also contradiction.

Lemma 2.3 Assume that (H_1) and (H_2) hold. Let $f_\infty = 0$, there exists λ^*, for any $\lambda = \lambda^*$, the problem(1) has at least one positive solution.

Proof Let $\{\lambda_n\}_{n=1}^\infty$ be a monotone decreasing sequence and $\lim\limits_{n \to \infty} \lambda_n = \lambda *$,and $\{\lambda_n\}_{n=1}^\infty$ satisfy the

problem(1), We claim that the corresponding positive solution sequence $\{y_{\lambda_n}(x)\}_{n=1}^\infty$ is uniformly bounded. Otherwise, there must be $\| y_{\lambda_n} \| = +\infty$.

Let $c_0 = \int_0^1 g(s)ds$, $R = \max\limits_{0 \leq y \leq M} f(x, y)$. In fact, from $\lim\limits_{y \to +\infty} \dfrac{f(x,y)}{y} = 0$, there exists $M > 0$, such that

$f(x,y) < \varepsilon y$, for all $y \geq M$, where ε is chosen such that $\varepsilon \lambda_0 \beta c_0 < 1$.

For all $y \geq 0$, $f(x,y) \leq R + \varepsilon y$. Hence we obtain

$$\left\| y_{\lambda_n}(x) \right\| = \lambda_n \max_{0 \leq x \leq 1} \int_0^1 G(x,s) f(s, y_{\lambda_n}(s))\mathrm{d}s$$

$$\leq \lambda_n \| \beta \| (R + \varepsilon \| y_{\lambda_n} \|) \int_0^1 g(s)\mathrm{d}s$$

such that $(1 - \lambda_n \| \beta \| c_0 \varepsilon) \| y_{\lambda_n} \| \leq \lambda_n \| \beta \| c_0 R$

$\| y_{\lambda_n} \| \leq \lambda_n \| \beta \| c_0 R / (1 - \lambda_n \| \beta \| c_0 \varepsilon)$

Let $n \to \infty$, it is obvious to see

$\lim\limits_{n \to \infty} \| y_{\lambda_n} \| \leq \lim\limits_{n \to \infty} \lambda_n \beta c_0 (R + \varepsilon \| y_{\lambda_n} \|) \leq \lambda_0 c_0 \| \beta \| R / (1 - \lambda_0 \| \beta \| c_0 \varepsilon) < +\infty$.

It is a contradict with $\| y_{\lambda_n} \| = +\infty$. Thus, there exists a number L with $0 < L < +\infty$ such that $\| y_{\lambda_n} \| \leq L$ for all n.

In addition, following from Φ is a completely continous mapping, we obtain that $\{y_{\lambda_n}\}_{n=1}^{\infty}$ is equicontinous. Ascoli-Arzela theorem claims that $\{y_{\lambda_n}\}_{n=1}^{\infty}$ has a uniformly convergent subsequence. Denoted again by $\{y_{\lambda_n}\}_{n=1}^{\infty}$. And $\{y_{\lambda_n}(x)\}_{n=1}^{\infty}$ converge to $y_0(x)$ uniformly on [0,1]. Thus, $y_0(x) = \lim_{n\to\infty} y_{\lambda_n}(x) \geq 0$, $y_{\lambda_n}(x)$ satisfies

$$y_{\lambda_n}(x) = \lambda_n \int_0^1 G(x,s) f(s, y_{\lambda_n}(s)) \mathrm{d}s$$

Let $n \to \infty$, using the Lebesgue dominated convergence theorem, we obtain

$$y_0(x) = \lim_{n\to\infty} y_{\lambda_n}(x) = \lim_{n\to\infty} \lambda_n \int_0^1 G(x,s) f(s, y_{\lambda_n}(s)) \mathrm{d}s = \lambda_0 \int_0^1 G(x,s) f(s, y(s)) \mathrm{d}s$$

Therefore, for any $\lambda = \lambda^*$, $y_0(x)$ is a positive solution of (1).

Lemma 2.4 Assume that (H_1) and (H_2) hold. Let $f_0 = 0$, $f_\infty = 0$, then there exists $\lambda_0 > 0$, such that for all $\lambda \in [\lambda_0, +\infty)$, then the problem(1) has at least one positive solution.

Proof Choose $\lambda_0 = \dfrac{1}{\|\beta\|}(m_0 \int_{\frac{1}{4}}^{\frac{3}{4}} g(s)\mathrm{d}s)^{-1}$, let $K_1 = \{y \in K : \|y\| < \dfrac{\alpha}{2\|\beta\|}\}$, then for

$y \in \partial K_1 = \{y \in K : \|y\| = \dfrac{\alpha}{2\|\beta\|}\}$, $\min_{\frac{1}{4} \leq x \leq \frac{3}{4}} y(x) \geq \alpha \|y\| / \|\beta\| = \dfrac{\alpha^2}{2\|\beta\|^2}$, $y \in \partial K_1$.

we denote $m_0 = \min_{\frac{\alpha^2}{2\|\beta\|^2} \leq x \leq \frac{\alpha}{2\|\beta\|}} f(x,y)$,

$$\|\Phi y(x)\| \geq \lambda_0 \alpha m_0 \int_{\frac{1}{4}}^{\frac{3}{4}} g(s)\mathrm{d}s = \dfrac{\alpha}{\|\beta\|} > \dfrac{\alpha}{2\|\beta\|} = \|y\|$$

Following from $\lim_{y\to 0^+} \dfrac{f(x,y)}{y} = 0$, for all $x \in [0,1]$ and $\varepsilon > 0$, there exists $0 < r < \dfrac{\alpha}{2\|\beta\|}$, when

$0 < y \leq r$, we can get $f(x,y) \leq \varepsilon_2 y$. where ε_2 is chosen such that $\lambda \varepsilon_2 \beta \int_0^1 g(s)\mathrm{d}s < 1$. For $y \in \partial k_r = \{y \in K_r : \|y\| = r\}$, we have

$$\|\Phi y(x)\| \leq \varepsilon_2 \lambda \beta \int_0^1 g(s)\mathrm{d}s \cdot \|y\| \leq \|y\|.$$

Hence, by Theorem 2.1, the problem (1) has at least one positive solution.

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