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Research Article

On sandwich theorems for certain analytic functions

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Abstract: In this paper, we derive some subordination and superordination results for certain normalized analytic functions in the open unit disk.
Keywords: univalent functions; starlike functions; subordination; superordination

INTRODUCTION

Let H denote the class of analytic functions in the open unit disc $U=\{z\in C:|z|<1\}$, and $H[a,n]$ denote the subclass of the functions $f\in H$ of the form

$$f(z)=a+a_n z^n+a_{n+1}z^{n+1}+\cdots. \tag{1}$$

Also,Let A be the subclass the functions $f\in H$ of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots. \tag{2}$$

A function $f\in A$ is said to be in the class S^* of starlike functions in U, if it satisfies the inequality $Re(\frac{zf'(z)}{f(z)})>0, z\in U$. Furthermore, a function $f\in A$ is said to be in the class C of convex functions in U, if it satisfies the inequality $Re(1+\frac{zf''(z)}{f'(z)})>0, z\in U$.

Let $f(z)$ and $F(z)$ be analytic in U, then we say that the function $f(z)$ is subordinate to $F(z)$ in U, if there exists an analytic function $w(z)$ in U such that $|w(z)|\le|z|$, and $f(z)\equiv F(w(z))$, denoted by $f\prec F$ or $f(z)\prec F(z)$. If $F(z)$ is univalent in U, then the subordination is equivalent to $f(0)=F(0)$ and $f(U)\subset F(U)$.

Let $p,h\in H$ and let $\varphi(r,s,t;z):C^3\times U\to C$. If p and $\varphi(p(z),zp'(z),z^2p''(z);z)$ are univalent and if p satisfies the second-order superordination

$$h(z)\prec \varphi(p(z),zp'(z),z^2p''(z);z), \tag{3}$$

then p is a solution of the differential superordination (3). (If f is subordinate to F, then F is superordinate to f.) An analytic function q is called a subordinant if $q\prec p$ for all p satisfying (3). A univalent subordinant Q that satisfies $q\prec Q$ for all subordinants q of (3) is said to be the best subordinant. Recently Miller and Mocanu [1] obtained conditions on h,q and φ for which the following implication holds:

$$h(z)\prec \varphi(p(z),zp'(z),z^2p''(z);z)\Rightarrow q(z)\prec p(z). \tag{4}$$

Using the results of Miller and Mocanu [1], Bulboacă [2] considered certain classes of first-order differential superordinations as well as superordination-preserving integral operators [3]. Ali et al. [4] have used the results of Bulboacă [2] and obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy

$$q(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z), \tag{5}$$

where q_1 and q_2 are given univalent functions in U with $q_1(0)=1$ and $q_2(0)=1$. Shanmugam et al. [5] obtained sufficient conditions for normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z) and q_1(z) \prec \frac{z^2 f'(z)}{f^2(z)} \prec q_2(z) \tag{6}$$

where q_1 and q_2 are given univalent functions in U with $q_1(0)=1$ and $q_2(0)=1$, while Obradović and Owa [6] obtained subordination results with the quantity $(f(z)/z)^{\mu}$ (see also [7]).

For $0<\alpha<1$, a function $f(z)\in N(\alpha)$ if and only if $f(z)\in A$ and

$$Re\{\frac{zf'(z)}{f(z)}(\frac{z}{f(z)})^{\alpha}\}>0, z\in U. \tag{7}$$

$N(\alpha)$ was introduced by M.Obradović [8] recently, and he called this class of functions to be non-Bazilevič type. Tuneski and Darus [9] obtained Fekete-Szegö inequality for the non-Bazileviˇc class of functions. Using this non-Bazilevič class, Wang et al. [10] studied many subordination results for the class $N(\alpha,\lambda,A,B)$ defined as

$$N(\alpha,\lambda,A,B)=\{f(z)\in A:(1+\lambda)(\frac{z}{f(z)})^{\alpha}-\lambda\frac{zf'(z)}{f(z)}(\frac{z}{f(z)})^{\alpha} \prec \frac{1+Az}{1+Bz}, z\in U\}. \tag{8}$$

where $0<\alpha<1, \lambda\in C, -1\le B\le 1, A\ne B, A\in R$.

The main object of the present sequel to the aforementioned works is to apply a method based on the differential subordination in order to derive several subordination results. Furthermore, we obtain the previous results of Srivastava and Lashin [7], Singh [11], Shanmugam et al. [12] and Obradović andOwa [6] as special cases of some of the results presented here.

Some lemmas

To prove our main result, we will need the following definition and lemmas:

DEFINITION

[1] Denote by Σ the set of all functions $f(z)$ that are analytic and injective on $\bar{U}-E(f)$, where

$$E(f)=\{\xi\in\partial U:\lim_{z\to\xi} f(z)=\infty\}, \tag{9}$$

and are such that $f'(\xi)\ne 0$ for $\xi\in\partial U-E(f)$.

Lemma

[5] Let q be univalent in U and let $\beta,\gamma\in C$ with $Re(1+\frac{zq''(z)}{q'(z)})>\max\{0,-Re\frac{\beta}{\gamma}\}$. If $p(z)$ is analytic in U and

$$\beta p(z)+\gamma zp'(z) \prec \beta q(z)+\gamma zq'(z), \tag{10}$$

then $p(z)\prec q(z)$ and q is the best dominant.

Lemma

[1] Let q be convex univalent in U and let $\gamma\in C$ with $Re(\gamma)>0$. If $p(z)\in H[q(0),1]\cap\Sigma$ and $p(z)+\gamma zp'(z)$ is univalent in U, and

$$q(z)+\gamma zq'(z) \prec p(z)+\gamma zp'(z), \tag{11}$$

then $q(z)\prec p(z)$ and q is the best subordinant.

Lemma

[13] Let q be univalent in U and let θ, ρ be analytic in a domain Ω containing $q(U)$ with $\rho(w) \neq 0$ when $w \in q(U)$. Set $h(z) = zq'(z)\rho(q(z))$, $F(z) = \theta(q(z)) + h(z)$. Suppose that

(1) $h(z)$ is starlike univalent in U;

(2) $Re(\frac{zF'(z)}{h(z)}) > 0$ for $z \in U$.

If

$$\theta(p(z)) + zp'(z)\rho(F(z)) \prec \theta(q(z)) + zq'(z)\rho(q(z)), \tag{12}$$

then $p(z) \prec q(z)$ and q(z) is the best dominant.

Lemma

[3] Let q be convex univalent in U, and let θ, ρ be analytic in a domain Ω containing $q(U)$. Suppose that

(1) $zq'(z)\rho(q(z))$ is starlike univalent in U;

(2) $Re(\frac{\theta'(q(z)}{\rho(q(z))}) > 0$ for $z \in U$.

If $p(z) \in H[q(0),1] \subseteq \Sigma$, with $p(U) \subset \Omega$ and $\theta(p(z)) + zp'(z)\rho(p(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\rho(q(z)) \prec \theta(p(z)) + zp'(z)\rho(p(z)), \tag{13}$$

then $q(z) \prec p(z)$ and q is the best subordinant.

Sandwich theroems

By using Lemma 2.1, we first prove the following Theorem.

Theorem 3.1. Let $q(z)$ be univalent in U, $0 < \lambda < 1$ and $\alpha \in C$. Further assume that

$$Re\{1 + \frac{zq''(z)}{q'(z)}\} > \max\{0, -Re\frac{\alpha}{\lambda}\}. \tag{14}$$

If $f(z) \in A, g(z) \in S^*$, then

$$(\frac{g(z)}{f(z)})^{\lambda} + \alpha z(\frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)})(\frac{g(z)}{f(z)})^{\lambda} \prec q(z) + \frac{\alpha}{\lambda} zq'(z), \tag{15}$$

implies that

$$(\frac{g(z)}{f(z)})^{\lambda} \prec q(z) \tag{16}$$

and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) = (\frac{g(z)}{f(z)})^{\lambda} \tag{17}$$

Then a computation shows that

$$(\frac{g(z)}{f(z)})^{\lambda} + \alpha z(\frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)})(\frac{g(z)}{f(z)})^{\lambda} = p(z) + \frac{\alpha}{\lambda} zp'(z). \tag{18}$$

Then we obtain that

$$p(z) + \frac{\gamma}{\alpha} zp'(z) \prec q(z) + \frac{\gamma}{\alpha} zq'(z). \tag{19}$$

By using Lemma 2.1,we have the result.

Theorem 3.2. Let $q(z)$ be convex univalent in U, $0<\lambda<1$, $\alpha \in C$ with $Re(\alpha)>0$. Suppose $(\frac{g(z)}{f(z)})^{\alpha} \in H[q(0),1] \cap \Sigma$ and

$$(\frac{g(z)}{f(z)})^{\lambda} + \alpha z(\frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)})(\frac{g(z)}{f(z)})^{\lambda} \tag{20}$$

be univalent in U. If

$$q(z) + \frac{\gamma}{\alpha} zq'(z) \prec (\frac{g(z)}{f(z)})^{\lambda} + \alpha z(\frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)})(\frac{g(z)}{f(z)})^{\lambda}, \tag{21}$$

then

$$q(z) \prec (\frac{g(z)}{f(z)})^{\alpha} \tag{22}$$

and $q(z)$ is the best subordinant.

Proof. Let $p(z) = (\frac{g(z)}{f(z)})^{\alpha}$. Then Theorem 3.2 follows by an application of Lemma 2.2.
Combining the results of differential subordination and superordination, we obtain the following sandwich result.

Corollary 3.3. Let $q_1(z)$ be univalent and let $q_2(z)$ be convex univalent in U, $0<\lambda<1$ and $\alpha \in C$ with $Re(\alpha)>0$. Suppose $q_2(z)$ satisfies (14). If $(\frac{g(z)}{f(z)})^{\lambda} \in H[q_1(0),1] \cap \Sigma$, $(\frac{g(z)}{f(z)})^{\lambda} + \alpha z(\frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)})(\frac{g(z)}{f(z)})^{\lambda}$ is univalent in U, and

$$q_1(z) + \frac{\alpha}{\lambda} zq_1(z) \prec (\frac{g(z)}{f(z)})^{\lambda} + \alpha z(\frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)})(\frac{g(z)}{f(z)})^{\lambda} \prec q_2(z) + \frac{\alpha}{\lambda} zq_2(z), \tag{23}$$

then

$$q_1(z) \prec (\frac{g(z)}{f(z)})^{\lambda} \prec q_2(z) \tag{24}$$

and $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and the best dominant.
For $\alpha = 1$ and $g(z) = z$, we get the following corollary.

Corollary 3.4. Let $q_1(z)$ be univalent and let $q_2(z)$ be convex univalent in U, $0<\lambda<1$. Suppose $q_2(z)$ satisfies (14). If $(\frac{g(z)}{f(z)})^{\lambda} \in H[q_1(0),1] \cap \Sigma$, $(2 - \frac{zf'(z)}{f(z)})(\frac{z}{f(z)})^{\lambda}$ is univalent in U, and

$$q_1(z) + \frac{1}{\alpha} zq_1(z) \prec (2 - \frac{zf'(z)}{f(z)})(\frac{z}{f(z)})^{\lambda} \prec q_2(z) + \frac{1}{\alpha} zq_2(z), \tag{25}$$

then

$$q_1(z) \prec (\frac{z}{f(z)})^{\alpha} \prec q_2(z) \tag{26}$$

and $q_1(z)$ and $q_2(z)$ are respectively the best subordinant and best dominant.

4. Open Problem

Let H be the class of analytic functions in $U = \{z \in C : |z| < 1\}$, and $H[a,p]$ be the subclass of H consisting of functions of the form

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots. \tag{27}$$

Let $A(p)$ be the subclass of H consisting of functions of the form

$$f(z) = z^p + a_{p+1} z^{p+1} + a_{p+2} z^{p+2} + \cdots. \tag{28}$$

A function $f \in A(p)$ is said to be in the class $S^*(p)$ of p-valent starlike functions in U, if it satisfies the inequality $Re(\frac{zf'(z)}{pf(z)}) > 0, z \in U$.

Let $f(z) \in A(p)$ and $g(z) \in S^*(p)$. We can consider sufficient conditions on h, q_1, q_2 and φ for which the following implication holds:

$$q_1(z) \prec (\frac{g(z)}{f(z)})^{\alpha} \prec q_2(z), \quad (29)$$

or

$$q_1(z) \prec (\frac{(1-\beta) f(z) + \beta z f'(z)}{g(z)})^{\alpha} \prec q_2(z), \quad (30)$$

where $0 < \alpha < 1$ and $0 \le \beta \le 1$.

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