

Research Article

Study on Constructing Mathematical model of City-Flood and Numerical Simulation

Jinzi Liu, Weijiao Di, Jiying Liu, Jing Wang

School of Mathematics and Statistics, Northeast Petroleum University, Daqing, China 163318

***Corresponding author**

Jinzi Liu

Email: jinzi19811216@126.com

Abstract: In this paper, it is suggested to natural phenomenon of rain-storm result in the accumulation disaster. Hypotheses proposed model based on physical background, variable selection coefficient convection - diffusion equation accurately described, and used the terrain function instead of convection - diffusion equation parameters term, comprehensive rainfall, drainage, infiltration and other related functions form a function matching the physical background. This article established mathematical model of urban flooding, using the finite element method, discussed related factors in detail. It could be combined with hydrodynamic and weather forecast. It could directly simulate and forecast the accumulation disaster and strengthen the ability for meteorological service. The importance of establishing the city-flood model is obvious.

Keywords: City flood; Convection-diffusion equations; Finite element

INTRODUCTION

Urban flooding (storm) is a special flood current problems facing China. Short and heavy rainfall caused waterlogging disasters have become increasingly prominent. It not only threatens the city's flood control and drainage tasks, but also on people's lives a greater impact. In recent years, heavy rain on the city more and more obvious hazards. Domestic only Beijing, for example, from 2009 to 2011, some sections of the city to form water, in some places the formation of congestion, part of the traffic to a standstill. In particular, in July 2012, 61 years is not a case of heavy rain led to Beijing Mountains as mudslides, the city suffered waterlogging disaster, the urban sections of water, traffic disruption, municipal water projects with multiple injuries, a large number of vehicles flooded, preliminary statistics economic losses million dollars. In addition to the assessable property, more concern is the number of casualties, as at 17:00 on the 22nd, were found in the territory due to heavy rainfall in Beijing killed 37 people.

For the past many years, it had lots of papers in field research of City-flood. C. G. Collier [1], published a paper on "Estimating Probable Maximum Precipitation Using a Storm Model Approach" whereby they used Math-theory on Storm Model. M. H. Hsu [2], presented analytical model of Urban Drainage System. Other scholar, Z. Christopher [3], Y. Sheng [4] worked on Urban Storm Water Models. G. Bekithemba [5], F. Daniel [6] studied on Rainfall-Runoff Model in Storm Drainage Network Design. Particularly, D. M. Li [7] researched theory and mathematical modeling of Urban Rainstorm Water Logging. However, it does not get more valuable research in the last decade.

Many experts and scholars at home and abroad establishment a series of mathematical and empirical models in the field of city flood. There are mainly three kinds of methods: Construction of gridded calculated by using nonlinear reservoir models, two dimensional unsteady hydraulic models, and analysis models of GIS. However, due to their characteristics, the model structure is complex, the computationally intensive, and time-consuming. Simple model structure is strongly dependent on the comprehensiveness of the information on GIS. Therefore, it needs to establish a practical application of city-flood model.

MATHEMATICAL MODEL

In this paper, City Flood refers to change of rainfall accumulation under normal operation of drainage system in city. Grades of rainfall intensity are light rain, moderate rain, heavy rain, and rainstorm. Rainstorm which refers to rainfall intensity is very big. Following two kinds of circumstances: rainfall exceeds 16mm/hour or 50mm/day.

This paper is assumption city-flood divided into two following rainstorm process.

Rainwater accumulation process

In the process of rainwater, drainage capacity of drainage and penetration system greater than rainfall, City streets will be short of water, if not a long time. The water can be safely discharged urban soil absorption and drainage system, will not cause harm.

Rainwater overflow process

I In the process of rainwater, rainfall greater than the drainage system of drainage capacity and penetration, urban soil (ground) absorption saturation, rainwater cannot be successfully discharged, but directly to the lower-lying areas around the overflow will cause relatively serious harm.

City-Flood Mathematical Model

Assumptions and Descriptions

Considering actual physical characteristics of city-flood, modify convection diffusion model construct mathematical model as follows:

$$k(x, y)\left(\frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2}\right) + \frac{\partial k(x, y)}{\partial x} \cdot \frac{\partial u(x, y, t)}{\partial x} + \frac{\partial k(x, y)}{\partial y} \cdot \frac{\partial u(x, y, t)}{\partial y} + \dots$$

$$+ F(x, y, t) + P(x, y, t) + S(x, y, t) = \frac{\partial u(x, y, t)}{\partial t}$$

Where $F(x, y, t)$ is Rainfall source function, $P(x, y, t)$ is Aggregation function of drainage system, $S(x, y, t)$ is Flow convergence function, $k(x, y)$ is diffusion coefficient, represented by topographic function. $\frac{\partial k(x, y)}{\partial x}$ and $\frac{\partial k(x, y)}{\partial y}$ are convection coefficient, $u(x, y, t)$ is water level. Boundary conditions: $u|_{\partial\Omega} = 0, t = 0$. Initial conditions: $u|_{\Omega} = 0, t = 0$.

Relevant parameters and Functions

Space step: $\Delta x = \Delta y = 1m$, time step: $\Delta t = 1min$, region grid dissection: 40*40, four region: $[0 \sim 20] \times [0 \sim 20], [0 \sim 20] \times [20 \sim 40], [20 \sim 40] \times [0 \sim 20], [20 \sim 40] \times [20 \sim 40]$, number of grid direction X is m, number of grid direction Y is n. The following diagram, levels represent the value, unit: M.

Considering the actual geographic shape, because the diffusion coefficient is a relationship with the terrain elevation and geology, and the same city geological difference not many (in the city green space and road building can be considered separately). So, we may assume that the optimization of banana function theory as the ground elevation function to represent the diffusion coefficient, banana function expressions are as follows:

$$Z(x, y) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

where $x_1 = [-2, 2]$, $x_2 = [-1, 3]$. The lowest point coordinates is (1,1). After meshing, the lowest point in the grid coordinates is (31,21).

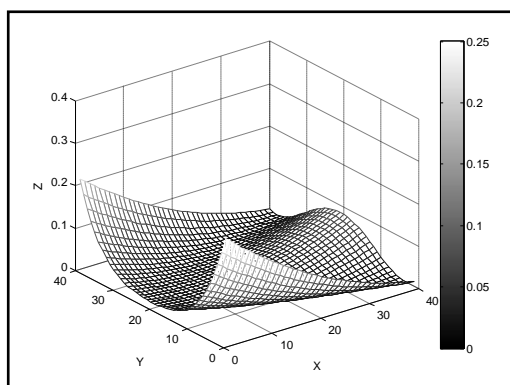


Fig-1: Topography map of city

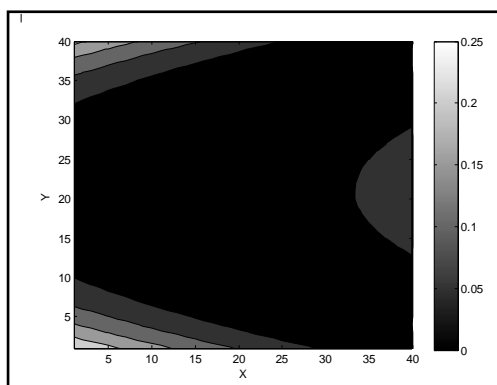


Fig-2: Contour of city's topography

Rainfall function hypothesis: The selection function $F(x, y, t)$ is used to simulate the space (x, y) of rainfall. Function expressions are as follows:

$$F(x, y, t) = F_0 \cdot e^{-a\sqrt{b(x-x_0)^2+(y-y_0)^2}}$$

where F_0 is rainfall of precipitation Center, a and m are coefficient, $F_0 = v \cdot t$, v is intensity of rainfall, assumed $v = 0.01m/min$. t is time, assumed $a = 0.02, b = 1.2$. The following represent rainfall contour distribution for 10~30 minutes. Grid split sorting 40*40.

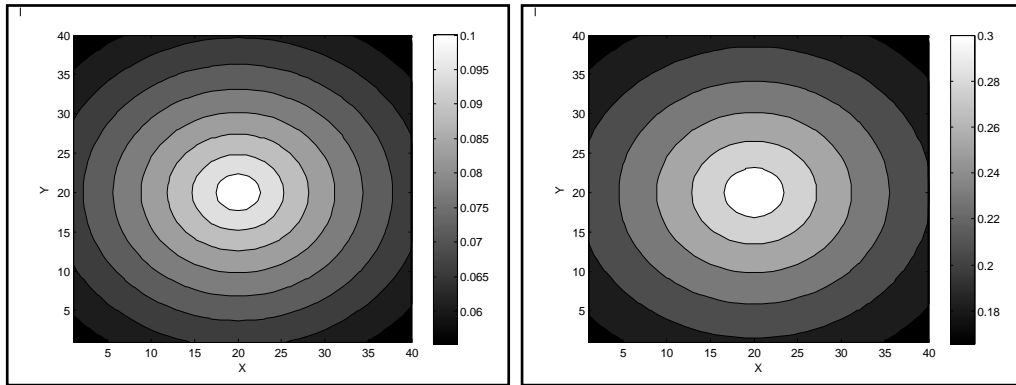


Fig-3: Contour of rainfall at ten minutes Fig-4: Contour of rainfall at thirty minutes

The drainage function hypothesis: hypothesis is the Aggregation function of drainage system, there is a relationship between structure and time and drainage network, simulation of drainage volume. This paper mainly described drainage capacity, as negative value. Function expressions are as follows:

$$P = \frac{1}{n_c} \cdot A_f \cdot R_f^{\frac{2}{3}} \cdot S_f^{\frac{1}{2}} \cdot t$$

where $n_c = 0.02$ is roughness coefficient, A_f is cross-sectional area of pipe. $R_f = 0.25m$ is hydraulic radius of pipeline. S_f is friction gradient, $S_f = \frac{n^2 \cdot u \sqrt{u^2 + v^2}}{d^{4/3}}$. n is ground roughness coefficient. d is depth from the ground. u, v are surface water flow rate.

Supposed $n = 0.04, d = 1.5m, u = v = 0.1m/s$, pipe diameter is 1m. It is average penetration in area except drain unit. Permeability function is $0.002t$, 20 minutes of saturated.

It has two drains. Lowest terrain as (31, 21) and (10, 10). With drainage depth reflects displacement, assumed saturation at 40 minutes. Following shows contour maps of displacement at 10~40min.

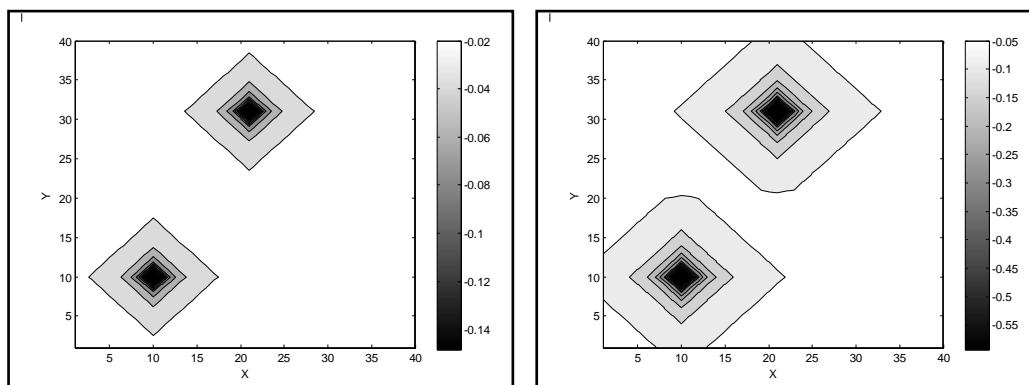


Fig-5: Contour of drain at ten minutes

Fig-6: Contour of drain at forty minutes

Special zone penetration function selection: permeability is not uniform in the region, for the rest of the region does not penetrate. Function:

$$S = \begin{cases} -0.005 \cdot t \cdot x / m, & x \in [0, 20] \times [0, 20] \\ 0 & , x \in \text{other area} \end{cases}$$

It assumed saturation after 20 minutes. Following shows contour maps of infiltration at 10~20min.

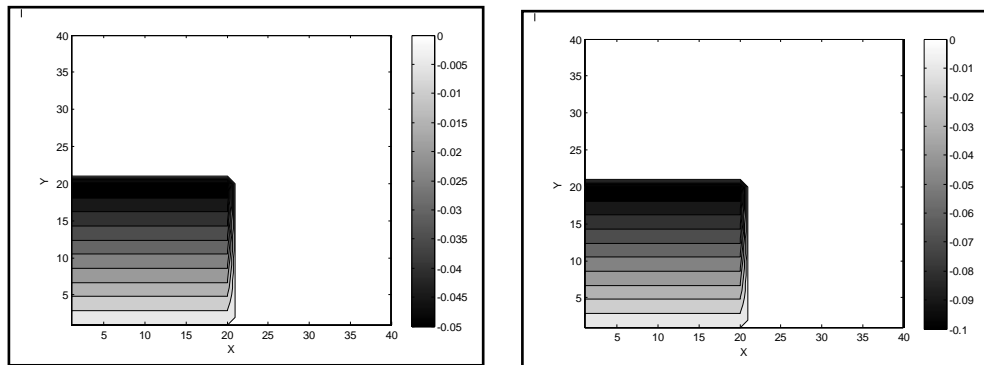


Fig-7: Contour of infiltrate at ten minutes Fig-8: Contour map of infiltrate at twenty minutes

Numerical Simulation

Definite solution problems of City-Flood mathematical model:

$$\frac{\partial u}{\partial t} - (k \frac{\partial^2 u}{\partial x^2} + k \frac{\partial^2 u}{\partial y^2} + \frac{\partial k}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial k}{\partial y} \frac{\partial u}{\partial y} + F + P + S) = 0 \quad (x, y) \in \Omega, t \in [0, T] \quad (1)$$

Initial condition

$$u = 0, (x, y) \in \Omega, t = 0 \quad (2)$$

Boundary condition

$$u = 0, (x, y) \in \partial\Omega, t \in [0, T] \quad (3)$$

where F, P, S are function of x, y, t independent variable, k is parameters.

Assumption

$$S_0^1 = \left\{ v \mid \iint_{\Omega} (v^2 + v_x^2 + v_y^2) dx dy \text{ 有意义}, v|_{\partial\Omega} = 0 \right\}$$

whereat $v \in S_0^1$, use v multiplied expression (1) again integral in addition use Green-Formula to solved as following formula:

$$\iint_{\Omega} \left[\frac{\partial u}{\partial t} v + k \frac{\partial u}{\partial x} \frac{\partial(v)}{\partial x} + k \frac{\partial u}{\partial y} \frac{\partial(v)}{\partial y} - \frac{\partial k}{\partial x} \frac{\partial u}{\partial x} v - \frac{\partial k}{\partial y} \frac{\partial u}{\partial y} v - (F + P + S)v \right] dx dy = 0 \quad (4)$$

Selection V_h as heuristic functional space is finite dimensional subspace of S_0^1 . notation V_h base as $\phi_1(x, y), \phi_2(x, y), \dots, \phi_n(x, y), V_h = span\{\phi_1, \phi_2, \dots, \phi_n\}$.

Expression (4) approximation form toward $\forall v_h \in V_h, t \in [0, T]$ written as

$$\iint_{\Omega} \left[\frac{\partial u_h}{\partial t} v_h + k_1 \frac{\partial u_h}{\partial x} \frac{\partial(v_h)}{\partial x} + k_1 \frac{\partial u_h}{\partial y} \frac{\partial(v_h)}{\partial y} - k_2 \frac{\partial u_h}{\partial x} v_h - k_2 \frac{\partial u_h}{\partial y} v_h - (F + P + S)v_h \right] dx dy = 0 \quad (5)$$

where, $u_h \in V_h$,

$$u_h(x, y, t) = \sum_{j=1}^n a_j(t) \phi_j(x, y) \quad (6)$$

Use time variable t as parameter, it contained in parameter $a_j(t)$ of base function expansion.

According to Galerkin method, choose $v_h = \phi_i(x, y)$ in expression (5), $i = 1, 2, \dots, n$, expression (6) substitute to expression (5), as follows

$$\sum_{j=1}^n [(\iint_{\Omega} \phi_i \phi_j dx dy) \frac{da_j(t)}{dt} + (\iint_{\Omega} (k \nabla \phi_j \nabla \phi_i) dx dy) a_j(t)] = \iint_{\Omega} (F + P + S) \phi_i dx dy \quad (7)$$

This is one linear system of ordinary differential equations of $a_j(t)$, $j = 1, 2, \dots, n$.

Solving equations, necessitate $a_0(t)$, $j = 1, 2, \dots, n$. Use expression (2), as follows

$$\sum_{j=1}^n a_j(0) \iint_{\Omega} \phi_i \phi_j dx dy = 0 \quad (8)$$

expression is linear equations of $\{a_j(0)\}$. Solved $a_j(0) = a_{j0}$, $j = 1, 2, \dots, n$. It can be obtained

$$u_h|_{t=0} = \sum_{j=1}^n a_j(0) \phi_j.$$

Equations (7) written in matrix form, as

$$\begin{aligned} A &= A(t) = [a_1(t), a_2(t), \dots, a_n(t)]^T \\ F &= F(t) = [\iint_{\Omega} (F + P + S) \phi_1 dx dy, \\ &\quad \iint_{\Omega} (F + P + S) \phi_2 dx dy, \\ &\quad \vdots \\ &\quad \iint_{\Omega} (F + P + S) \phi_n dx dy]^T \\ M &= [m_{ij}] \\ L &= [l_{ij}] \\ K &= [k_{ij}] \end{aligned}$$

where

$$\begin{aligned} m_{ij} &= \iint_{\Omega} \phi_i \phi_j dx dy \\ l_{ij} &= \iint_{\Omega} (k \phi_j (\frac{\partial \phi_i}{\partial x} + \frac{\partial \phi_i}{\partial y})) dx dy \\ k_{ij} &= \iint_{\Omega} (\phi_j \frac{\partial k}{\partial x} \frac{\partial \phi_i}{\partial x} + \phi_j \frac{\partial k}{\partial y} \frac{\partial \phi_i}{\partial y}) dx dy \end{aligned}$$

It obtained differential equations of A

$$M \frac{dA}{dt} + (K + L)A = F \quad (9)$$

Added on $a(0)$ initial condition

$$A(0) = [a_{10}(t), a_{20}(t), \dots, a_{n0}(t)]^T \quad (10)$$

It can solved $A(t)$.

Assumption time step $\Delta t, t_m = m\Delta t$, $\Delta t = \frac{T}{N}$, $m = 0, 1, \dots, N$. Between the time layer m and $m+1$ layer,

$t_m \leq t \leq t_{m+1}$, convert t to τ , $t = (m + \tau)\Delta t$, Then $0 \leq \tau \leq 1$, according to $a_j(t)$ linear interpolation and

$$a_j(t) = (1 - \tau)a_j(m\Delta t) + \tau a_j((m+1)\Delta t)$$

Then

$$\frac{da_j(t)}{dt} = \frac{1}{\Delta t} [-a_j(m\Delta t) + a_j((m+1)\Delta t)]$$

So, it obtain $a_j(t)$ is Component of vector $A(t)$ interpolation

$$A(t) = (1 - \tau)A_m + \tau A_{m+1} \tag{11}$$

$$\frac{dA}{dt} = \frac{1}{\Delta t} (A_{m+1} - A_m) \tag{12}$$

Where $A_m = [a_1(m\Delta t), a_2(m\Delta t), \dots, a_n(m\Delta t)]^T$, similar to A_{m+1} .

Similarly, the same way to vector F linear interpolation it obtained

$$F = (1 - \tau)F_m + \tau F_{m+1} \tag{13}$$

Where $F_m = [\iint_{\Omega} F|_{t=t_m} \phi_1 dx dy, \dots, \iint_{\Omega} F|_{t=t_m} \phi_n dx dy]^T$, similar to F_{m+1} . Above expressions (9) transform as following

$$\frac{1}{\Delta t} M(A_{m+1} - A_m) + (K + L)[(1 - \tau)A_m + \tau A_{m+1}] = (1 - \tau)F_m + \tau F_{m+1} \tag{14}$$

If expressions (14) multiplied non negative weight function $w(\tau)$, then Integral τ from 0 to 1, it obtains

$$\begin{aligned} & \frac{1}{\Delta t} M(A_{m+1} - A_m) \int_0^1 w(\tau) d\tau + (K + L) [A_m \int_0^1 w(\tau)(1 - \tau) d\tau + A_{m+1} \int_0^1 w(\tau)\tau d\tau] \\ & = F_m \int_0^1 w(\tau)(1 - \tau) d\tau + F_{m+1} \int_0^1 w(\tau)\tau d\tau \end{aligned}$$

Above expressions divide $\int_0^1 w(\tau)\tau d\tau$, and $\theta = \frac{\int_0^1 w(\tau)\tau d\tau}{\int_0^1 w(\tau) d\tau}$, it obtains

$$\left(\frac{1}{\Delta t} M + \theta(K + L)\right)A_{m+1} + \left(-\frac{1}{\Delta t} M + (1 - \theta)(K + L)\right)A_m = \theta F_{m+1} + (1 - \theta)F_m \tag{15}$$

If given vector F , it can computer F_{m+1} , F_m . If it has already obtained the approximation A_m , expressions (15) A_{m+1} can soled. such from $A_0 = A(0)$, it stratified value A .

This paper select weight function $w(\tau)$ as function $\delta(\tau = \frac{1}{2})$, it calculated $\theta = \frac{1}{2}$. Such as above process can be solved.

Numerical Examples

Space step: $\Delta x = \Delta y = 1m$, time step: $\Delta t = 1 \text{ min}$, region grid dissection: 40*40, four region: $[0 \sim 20] \times [0 \sim 20]$, $[0 \sim 20] \times [20 \sim 40]$, $[20 \sim 40] \times [0 \sim 20]$, $[20 \sim 40] \times [20 \sim 40]$, number of grid direction X is m number of grid direction Y is n. The water level is relative to the ground height, unit: M.

The first situation, drainage and infiltration systems are normal. Contour of water level on time series as following:

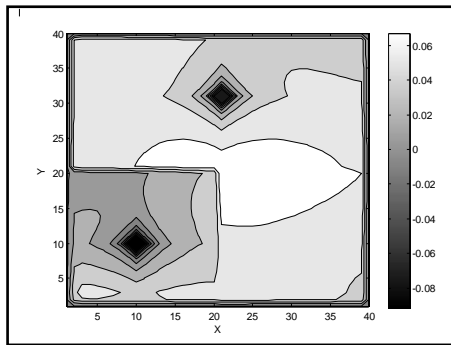


Fig-9: Contour of water level at ten minutes

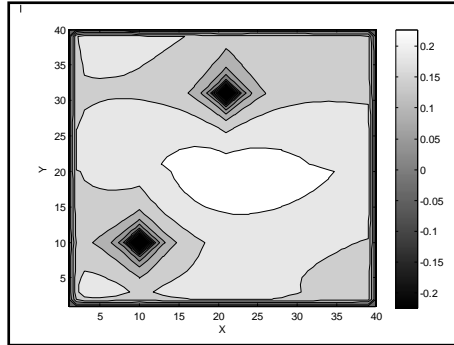


Fig-1:0 Contour of water level at thirty minutes

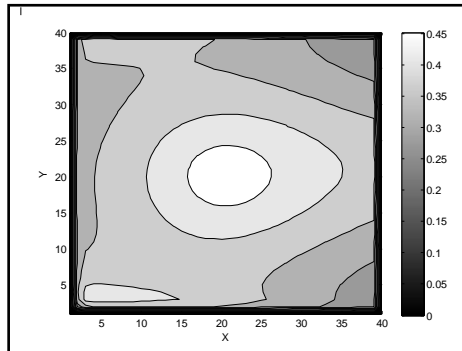


Fig-11: Contour of water level at fifty minutes

When the drainage system and infiltration system are normal, 10-20 minutes from the map can see obvious drainage and seepage phenomenon, the water level is smaller, less water. If at this time the rain stopped, ground water can be quickly discharged. After 20-40 minutes of permeability, from the graph has no traces of infiltration, but this time the normal drainage, water more. After 40-50 minutes, the drainage to the limit, a lot of water, forms a water logging.

The second situation, infiltration systems aren't work and drainage are normal work. Contour of water level on time series as following:

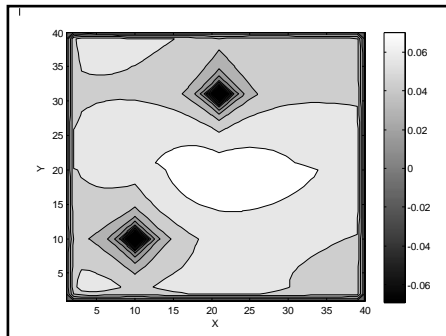


Fig-12: Contour of water level at ten minutes

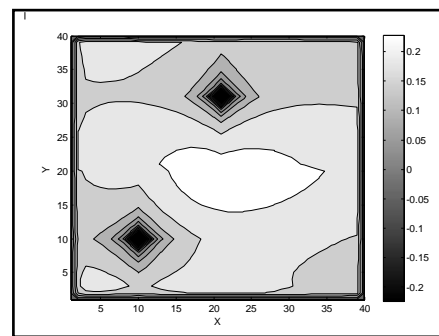


Fig-13: Contour of water level at thirty minutes

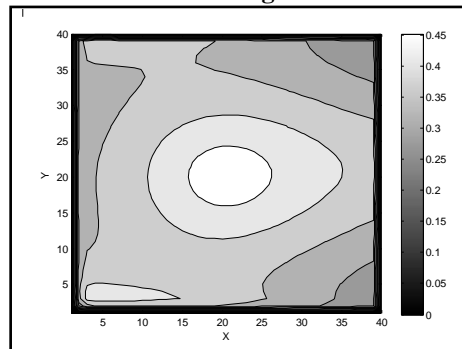


Fig-1:4 Contour of water level at fifty minutes

When the drainage system is normal, 10-20 minutes from the map can see that drain phenomenon, the water level is smaller, less water. If at this time the rain stopped, ground water can be quickly discharged. After 20~40 minutes, but this time the normal drainage, water more. After 40-50 minutes, the drainage to the limit, a lot of water, forms a waterlogging.

The third situation, drainage aren't work and infiltration systems are normal work. Contour of water level on time series as following:

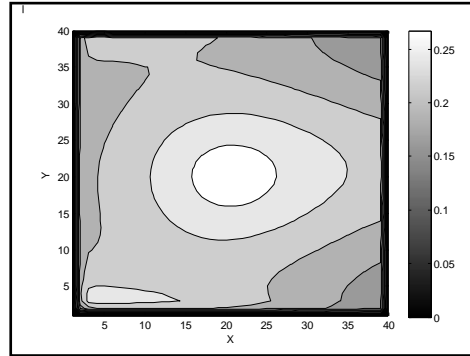
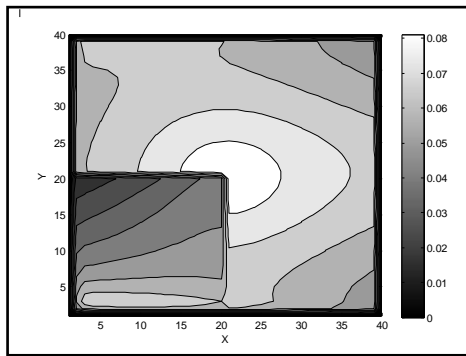


Fig-15: Contour of water level at ten minutes Fig-16: Contour of water level at thirty minutes

When the osmotic system is normal, 10-20 minutes from the map can see the phenomenon of osmosis, the water level is smaller, less water. If the rain stops, surface water is not easy to water logging. After 20 minutes, permeability, water more, difficult to discharge.

The fourth situation, drainage and infiltration systems aren't work normally. Contour of water level on time series as following:

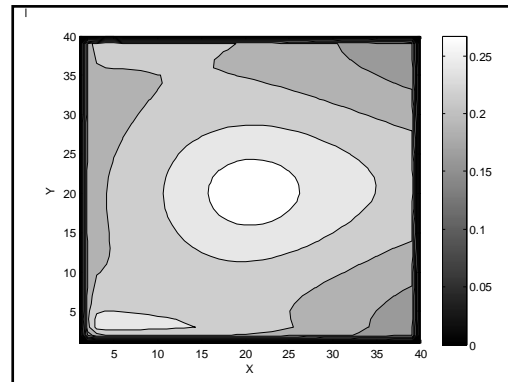
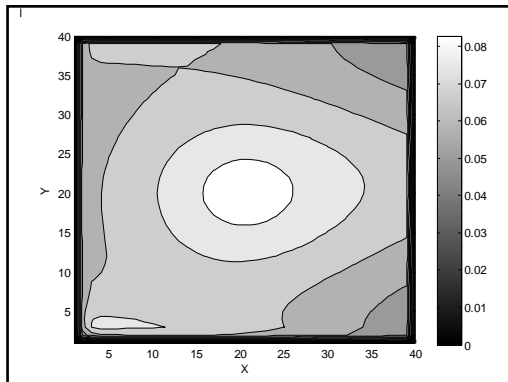


Fig-17: Contour of water level at ten minutes Fig-18: Contour of water level at thirty minutes

When the penetration systems and drainage systems are out of work, 10-20 minutes can be seen from the figure, the phenomenon of water, less water, less water. If at this time the rain stopped, an area of water can be quickly discharged. After 20 minutes, the water more difficult to discharge, the formation of water logging.

RESULTS AND DISCUSSION

This article elaborately introduced the theory and actuality of city flood. And it clarifies the process of rectified model: replace terrain function to diffusion parameter, integrate rainfall function, and drain function and seepage function to special function. Then it chooses finite-element method to solve convection-diffusion equations. It applied to this model. Apparently, this model could be exactly described city flood. The inverse method is feasible in the process of simulation. For the further exaltation level and the ability of meteorological service, it needs to form one accurate and real-time warning system. It could be the combination of hydrodynamic and weather forecast. It could directly simulate and forecast the accumulation disaster and strengthen the ability for meteorological service. The importance of constructing the city-flood model is obvious.

ACKNOWLEDGEMENTS

This work was supported by a grant from Young Natural Science Foundation of Northeast Petroleum University (Grant No.NEPUQN2014-22) .

REFERENCES

1. Collierl CG, Hardaker PJ; Estimating Probable Maximum Precipitation Using A Storm Model Approach. *Journal of Hydrology*, 1996; 183(3-4):277-306.
2. Hsu MH, Chen SH, Chang TJ; Inundation Simulation for Urban Drainage Basin with Storm Sewer System. *Journal of Hydrology*, 2000; 234 (1-2):21-37.
3. Christopher Z; Review of Urban Storm Water Models. *Environmental Modelling & Software*, 2001; 16(3):195-231.
4. Sheng Y; The Gumbel Logistic Model for Representing a Multivariate Storm Event. *Advances in Water Resources*, 2001; 24(2):179-185.
5. Bekithemba G, Nelson M, George S, Hubert HG; Coupling of Digital Elevation Model and Rainfall-Runoff Model in Storm Drainage Network Design. *Physics and Chemistry of the Earth*, 2002; 27(11-22):755-764.
6. Daniel F, James SA,Paul D; Bates. Attenuating Reaches and the Regional Flood Response of an Urbanizing Drainage Basin. *Advances in Water Resources*, 2003; 26(6):673-684.
7. Li DM, Zhang HP; Basic Theory and Mathematical Modeling of Urban Rainstorm Water Logging. *Journal of Hydra mechanics*, Ser. B, 2004; 16(1):17-27