Scholars Journal of Engineering and Technology (SJET) **ISSN 2321-435X (Online)**
Sch. J. Eng. Tech., 2015; 3(9):732-736 **ISSN 2347-9523 (Print)**
©Scholars Academic and Scientific Publisher
(An International Publisher for Academic and Scientific Resources)
www.saspublisher.com

Research Article

Rectangular group congruences on an epigroup

Yingqin Jin
Department of Computer and Technology, Anhui University of Science and Technology,
Huainan, Anhui 232001, P. R. China

***Corresponding author**
Yingqin Jin
Email: yqjin@aust.edu.cn

Abstract: An epigroup is a semigroup in which some power of any element lies in a sub-group of the given semigroup. The rectangular group congruences on an epigroup are investigated. A characterization of rectangular group congruences on an epigroup in terms of its rectangular group congruence pairs is given. Moreover, it is proved that the rectangular group congruence on an epigroup is uniquely determined by its kernel and hyper-trace.
Keywords: epigroups; Rectangular group congruences; Kernels, Hyper-traces.
MR(2000) Subject Classification : 20M10.

INTRODUCTION

Congruences on semigroups have been the subjects of continued investigation for many years. In particular, congruences on a regular semigroup have been explored extensively. It is well known that there are two principal approaches to study congruences on a regular semigroup: the kernel trace approach and the kernel normal system one. Preston [1] introduced the kernel normal system for congruences and characterized the congruences on inverse semigroups by means of it. Pastijn and Petrich [2] described the congruences on regular semigroups by their kernels and traces. In addition, the concept of congruence pairs is another effective tool for handing congruences on regular semigroups. Gomes extended the kernel trace approach to the kernel hyper-trace one and used it to describe the R-unipotent congruences and the orthodox congruences on a regular semigroup in [3, 4]. Moreover, Tan [5] described the rectangular group congruences on a regular semigroup by means of the kernel hyper-trace approach. Using the weak inverses in semigroups, Luo [6, 7] generalized the corresponding results for regular semigroups to eventually regular semigroups.

The aim of this paper is to establish analogues to the results in [5]. We prove that the rectangular group congruence on an epigroup is uniquely determined by its kernel and hyper-trace. Furthermore, we give an abstract characterization of rectangular group congruences by means of the rectangular group congruence pairs.

PRELIMINARIES

Throughout this paper, we follow the notation and conventions of Howie [8].

Let S be a semigroup and a in S. As usual, E_S is the set of all idempotents of S, $<E_S>$ is the subsemigroup of S generated by E_S, $\operatorname{Re}gS$ is the set of all regular elements of S and $V(a)$ = $\{a \in S | a = axa, x = xax\}$ is the set of all inverses of a. An element x of S is called a weak inverse of a if $xax = x$. Denote by $W(a)$ the set of all weak inverses of a in S. A semigroup S is called rectangular group if it is a regular semigroup whose idempotents form a rectangular band. In fact, rectangular groups are orthodox completely simple semigroups.

Let ρ be a congruence on a semigroup S. The restriction of ρ to the set E_S is called the trace of ρ denoted by $tr\rho$. Furthermore, the restriction of ρ to the subsemigroup $<E_S>$ is called the hyper-trace of ρ denoted by $htr\rho$. The subset $\{a \in S | a\rho \in E(S/\rho)\}$ of S is called the kernel of ρ denoted by $\ker \rho$. Recall that a congruence ρ on a semigroup S is said to be regular congruence if S/ρ is a regular semigroup. In particular, a congruence ρ on a semigroup S is said to be rectangular group congruence if S/ρ is a rectangular group.

Let S be a semigroup and e, f in E_S. Define $M(e,f)=\{g\in E_S | ge=g=fg\}$ and $S(e,f)=\{g\in E_S | ge=g=fg, egf=ef\}$. $S(e,f)$ is called the sandwich set of e and f. As is well known that the set $M(e,f)$ is nonempty for all $e,f\in E_S$ in an eventually regular semigroup. The set $M(e,f)$ will play an important role in eventually regular semigroups as the set $S(e,f)$ in regular ones.

The following concepts will play fundamental roles in this paper.

Definition 2.1 A subsemigroup K of S is said to be normal if

(1) $a\in K, a'\in W(a)\Rightarrow a'\in K$;

(2) $E_S\subseteq K$;

(3) $a\in S, a'\in W(a)\Rightarrow aKa', a'Ka\subseteq K$.

Definition 2.2 A congruence ε on the subsemigroup $<E_S>$ of S is said to be normal if for all $x,y\in <E_S>$, $a\in S$ and $a'\in W(a)$, then

$x\varepsilon y\Rightarrow axa'\,\varepsilon\, aya', a'xa\,\varepsilon\, a'ya$ whenever $axa', aya', a'xa, a'ya\in <E_S>$.

Definition 2.3 Let ε be a normal congruence on $<E_S>$ such that $<E_S>/\varepsilon$ is a rectangular band and K be a normal subsemigroup of S. Then a pair（ε, K）is said to be a rectangular group congruence pair of S, if it satisfies:

(RCP1) $\forall a\in S, a'\in W(a)$, there exists $a''\in W(a)$ such that $aa''aa'\,\varepsilon\, aa'$, $a'aa''a\,\varepsilon\, a'a$;

(RCP2) $\forall a\in S, x\in <E_S>, xa\in K\Rightarrow a\in K$.

Giving such a pair（ε, K）, we define a binary relation $\rho_{(\varepsilon,K)}$ on S by:

$$a\rho_{(\varepsilon,K)}b \Leftrightarrow \begin{cases}\forall a'\in W(a), \exists b'\in W(b), a'b\in K, aa'\,\varepsilon\, bb', a'a\,\varepsilon\, b'b\\ \forall b'\in W(b), \exists a'\in W(a), b'a\in K, aa'\,\varepsilon\, bb', aa'\,\varepsilon\, b'b\end{cases}$$

We denote by $RC(S)$ the set of all rectangular group congruences on S and denote by $RCP(S)$ the set of all rectangular group congruence pairs of S. In the sequel, let $\rho=\rho_{(\varepsilon,K)}$ is a rectangular group congruence pair of S in order to simplify the notation.

The following lemmas give some properties of an eventually regular semigroup, which will be used in Section 3. Since the case of epigroups is the special case of eventually regular semigroups, the results on eventually regular semigroups are also true for epigroups.

Lemma 2.4 [6] Let S be an eventually regular semigroup. If $a,b\in S, a'\in W(a), b'\in W(b)$ and $g\in M(a'a, bb')$, then $b'ga'\in W(ab)$.

Lemma 2.5 [9, 3] Let S be an eventually regular semigroup and ρ be a congruence on S.

(1) If $b\rho\in W(a\rho), a,b\in S$, then there exists $a'\in W(a)$ such that $a'\rho b$;

(2) If $e\rho f$ for some $e,f\in E_S$, then there exists $g\in M(e,f)$ such that $e\rho g\rho f$;

(3) If $a\rho$ is an idempotent of S/ρ, then an idempotent e can be found in such that $a\rho e$.

Lemma 2.6 [10] A congruence ρ on an eventually regular semigroup S is regular if and only if for all $a\in S$, there exists $a'\in W(a)$ such that $a\rho aa'a$, where $aa', a'a\in E_S$.

Lemma 2.7 [6] Let S be an eventually regular semigroup and $x\in <E_S>$, then $W(x)\subseteq <E_S>$.

3. Main Results

We begin the section with the main result of this paper.

Theorem 3.1 Let S be an epigroup. If $(\varepsilon, K)\in RCP(S)$, then $\rho_{(\varepsilon,K)}\in RC(S)$ is the unique rectangular group congruence on S such that $\ker\rho_{(\varepsilon,K)}=K$ and $htr\rho_{(\varepsilon,K)}=\varepsilon$.

We follow the proof of Theorem 3.1 by a series of lemmas, where S always represents an epigroup without extra illustration.

Lemma 3.2 Let $(\varepsilon, K) \in RCP(S)$ and $a,b \in K$. If $ab \in K$, then $axb \in K$ for all $x \in < E_S >$.

Proof. Let $ab \in K$ for some $a,b \in S$. Then $a'aba \in K$ for all $a' \in W(a)$. By the condition (RCP2), we get $ba \in K$ and $a'a \in < E_S >$. Therefore

$$(ba)x \in K, \quad b'baxb \in K \text{ for any } x \in < E_S >, b' \in W(b).$$

It follows from the condition (RCP2) that $axb \in K$.

Lemma 3.3 Let $(\varepsilon, K) \in RCP(S)$. Then ρ is a congruence on S.

Proof. We first show that ρ is an equivalence on S. It is obvious that ρ is symmetry. The fact that ρ is reflexive follows from $E_S \subseteq K$ and ε is reflexive. To show that ρ is transitive, let a ρ b and b ρ c for some $a,b,c \in S$. Then for $a' \in W(a)$, there exists $b'' \in W(b)$ such that $a'b \in K, aa' \ \varepsilon \ bb'', a'a \ \varepsilon b''b$, and for $b' \in W(b)$, there exists $c' \in W(c)$ such that $b'c \in K, \ bb' \ \varepsilon \ cc', \ b'b \ \varepsilon \ c'c$. Therefore $aa' \ \varepsilon \ cc', a'a \ \varepsilon \ c'c$ since ε is transitive. Put $g \in M(aa', bb')$. Then $b'ga \in W(a'b) \subseteq K$ since K is normal and $a'b \in K$. Moreover, we may obtain $a'bb'ga \in E_s \subseteq K, a'bb'c \in K$ and $gaa' \in < E_S >$. It follows from Lemma 3.2 that $a'bb'(gaa')c \in K$, so that $a'c \in K$ by (RCP2). Dually, we may show that for all $c' \in W(c)$, there exists $a' \in W(a)$ such that $c'a \in K, aa' \ \varepsilon \ cc', a'a \ \varepsilon \ c'c$, and so a ρ c, which implies that ρ is transitive. Consequently, ρ is an equivalence on S.

We now show that ρ is a congruence on S. To show that ρ is left compatible, Let a ρ b with some $a,b \in S$. For any $(ca)' \in W(ca)$, then

$$a' = (ca)'c \in W(a), c' = a(ca)' \in W(c), (ca)' = a'c', aa' = c'c.$$

It follows from a ρ b that $(ca)'cb = a'b \in K$. And since there exists $b' \in W(b)$ such that $aa' \ \varepsilon \ bb'$ by the definition of ρ, it follows from Lemma 2.5 that there exists $g \in M(aa', bb') = M(c'c, bb')$ such that $c'c = aa'$ and $aa' \ \varepsilon \ g \ \varepsilon \ bb'$. Put $(cb)' = b'gc'$. Then $(cb)' = b'gc' \in W(cb)$, which gives that

$$\begin{aligned} (ca)(ca)' &= caa'c' \\ &\varepsilon \ c(bb'g)c' \\ &= cb(cb)' \end{aligned} \quad \text{and}$$

$$\begin{aligned} (cb)'(cb) &= b'gc' = b'gb \\ &\varepsilon \ b'b \quad (g \ \varepsilon \ bb' \ \varepsilon \ aa') \\ &\varepsilon \ aa' = a'aa'a = a'c'ca = (ca)'(ca). \end{aligned}$$

A similar argument will show that for any $(cb)' \in W(cb)$, there exists $(ca)'' \in W(ca)$ such that $(cb)'ca \in K, ca(ca)'' \ \varepsilon \ cb(cb)', (ca)''ca \ \varepsilon \ (cb)'cb$. Hence ca ρ cb, together with the fact that ρ is an equivalence, and so ρ is a left congruenceon S.

On the other hand , the assertion that ρ is a right congruence on S can be shown dually. Thus ρ is a congruence on S, as required.

We now establish the connection between a rectangular group congruence and its kernel and hyper-trace.

Lemma 3.4 Let $(\varepsilon, K) \in RCP(S)$. Then $htr\rho = \varepsilon$ and ρ is a rectangular group congruence on S.

Proof. To prove that $htr\rho = \varepsilon$, let $x htr\rho y$ for some $x, y \in < E_S >$. Since ε is a rectangular band congruence on $< E_S >$, we, by Lemma 2.7, conclude that there exist $x' \in W(x) \bigcap < E_S >, y' \in W(y) \bigcap < E_S >$ such that $x \ \varepsilon \ xx'x, xx' \ \varepsilon \ yy', x'x \ \varepsilon \ y'y$.

Hence $x \,\varepsilon\, xx'x \,\varepsilon\, yy'x \,\varepsilon\, yx$ and

$$y \,\varepsilon\, yy'y \text{ (since } \varepsilon \text{ is a rectangular band congruence on } < E_S >)$$
$$\varepsilon\, yx'x$$
$$\varepsilon\, yx,$$

and so $x \,\varepsilon\, y$, which leads to $htr\rho \subseteq \varepsilon$.

Conversely, let $x \,\varepsilon\, y$ for some $x, y \in < E_S >$. Assume that $x' \in W(x) \bigcap < E_S >$. Then $x'y \in < E_S > \subseteq K$, and so $x'yx' \,\varepsilon\, x'x \,\varepsilon\, x'$ from the fact that ε is a rectangular band congruence on $< E_S >$. Hence $x'\,\varepsilon \in W(y\varepsilon)$, and so there exists $y' \in W(y) \bigcap < E_S >$ such that $x' \,\varepsilon\, y'$ by Lemma 2.5. It follows that $xx' \,\varepsilon\, yy', x'x \,\varepsilon\, y'y$. Furthermore, for $x' \in W(x)$, there exists $x' \in W(x)$ such that $y'x \in K, xx' \,\varepsilon\, yy', x'x \,\varepsilon\, y'y$, and so x ρ y, which leads to $\varepsilon \subseteq htr\rho$. Thus we deduce that $htr\rho = \varepsilon$, as required.

We now show that $\rho = \rho_{(\varepsilon, K)}$ is a regular congruence on S. For $a \in S, a' \in W(a)$, there exists $a'' \in W(a)$ such that

$$a'aa''a \in K, \quad a'aa''aa'a \,\varepsilon\, a'aa'a = a'a$$

so that $(a'a)\,\varepsilon \in W((a''a)\varepsilon)$. It follows from Lemmas 2.5 and 2.7 that there exists $(a''a)' \in W(a''a) \bigcap < E_S >$ such that $(a''a)' \,\varepsilon\, a'a$. And since

$$(a''a)'(aa''a)(a''a)'a' \rho (a''a)'(a''a)'a''a(a''a)'a'$$
$$= (a''a)'(a''a)'a'$$

$\rho (a''a)'a'$ (Since $< E_S >$ is a rectangular band and $htr\rho = \varepsilon$), which implies that $((a''a)'a')\rho \in W((aa''a)\rho)$. It follows from Lemma 2.5 that there exists $(aa''a)' \in W(aa''a)$ such that $(aa''a)' \rho (a''a)'a'$, and so

$$aa''a(aa''a)' \,\rho\, aa''a(a''a)'a'$$
$$\rho\, aa''aa'aa'$$
$$= aa''aa' \,\varepsilon\, aa', (由 RCP1)$$

which leads to $aa''a(aa''a)' \,\varepsilon\, aa''aa' \,\varepsilon\, aa'$ by $htr\rho = \varepsilon$. Furthermore, we obtain

$$(aa''a)'aa''a \,\rho\, (a''a)'a'aa''a$$
$$\rho a'aa'aa''a$$
$$= a'aa''a$$
$$\varepsilon\, a'a, \quad (由 RCP1)$$

and so $(aa''a)'aa''a \,\varepsilon\, a'aa''a \,\varepsilon\, a'a$ by $htr\rho = \varepsilon$. For $c \in W(aa''a)$, then $ca \in W(a''a)$, $ac \in W(aa'') \in < E_S > \subseteq K$, $caa''a(ca)caa''a \,\varepsilon\, caa''a$, and so $(caa''a)\,\varepsilon \in W((ca)\varepsilon)$. Hence there exists $x \in W(ca) \bigcap < E_S >$ such that $x \,\varepsilon\, caa''a$. Put $a' = xc$. Then $a' \in W(a)$, so that $a(xc) \,\varepsilon\, aa''axc$ $\varepsilon\, aa''acaa''ac \,\varepsilon\, (aa''a)c$ and $(xc)a \,\varepsilon\, c(aa''a)$. Hence $a\rho aa''a$, and so it follows from Lemma 2.6 that ρ is a regular congruence on S.

To show that S is a rectangular group congruence on S, let $a\rho, b\rho \in E(S/\rho)$. Then there exist $e, f \in E_S$ such that $a\rho e, bef$, and so $efe\rho e$ from the fact that $< E_S >/\varepsilon$ is a rectangular band. Thus $(aba)\rho a$, and so S/ρ is a rectangular group, which gives that ρ is a rectangular group congruence on S.

Lemma 3.5 Let $(\varepsilon, K) \in RCP(S)$. Then $\ker \rho = K$.

Proof. To prove $\ker \rho = K$, let $a \in \ker \rho$ for some $a \in \mathrm{S}$. Then there exists $e \in E_S$ such that $a\rho e$, and so there exists $e' \in W(e) \bigcap < E_S >$ such that $e'a \in K$. Hence $a \in K$ by (RCP2), so that $\ker \rho \subseteq K$.

Conversely, if $a \in K$, then $a' \in W(a) \subseteq K$ since K is a normal subsemigroup of S, and so $a'a^2 \in K$. Suppose $e \in M(a'a, aa')$. It follows from Lemma 2.4 that $a'ea' \in W(a^2)$. And since ρ is a regular congruence on S, together with Lemma 2.7, there exists $a'' \in W(a)$ such that $a\rho aa''a$. Hence

$$(a'ea')a^2 \ \rho \ a'a(a'ea'aa)a''a$$

$$\rho \ a'aa''a \qquad \text{(Since } \rho \text{ is a rectangular group congruence)}$$

$$\rho a'a,$$

where $a'ea'a^2, a'a, a''a \in E_S$, so that $a'ea'a^2 \ \varepsilon \ a'a$ by $htr\rho = \varepsilon$. Notice that

$$a^2(a'ea') \ \rho \ aa''(aaa'ea')aa'$$

$$\rho \ aa''aa' \qquad \text{(Since } \rho \text{ is a rectangular group congruence)}$$

$$\rho \ aa'$$

where $a^2a'ea', a'a \in E_S$, so that $a^2(a'ea') \ \varepsilon \ aa'$ by $htr\rho = \varepsilon$.

On the other hand, let $a' = ac, a''' = ca$ for any $c \in W(a^2)$. Then $a', a''' \in W(a)$ and $c = a'''a', a'a = aa'''$, and so $ca = a''' \in K$ since $a \in K$ and K is a normal subsemigroup of S. Hence we ,since $< E_S >/\varepsilon$ is a rectangular band , obtain $a''a(a'''aa''a)a'a \ \varepsilon \ a''aa'a$ and $a''a(aa''')a'a \ \varepsilon \ (a''a)a'a$, and so $a''a(a'''aa''a)a'a \ \varepsilon \ a''a(aa''')a'a$ (1)

It follows from $aa'a'aaa' \ \varepsilon \ aa'$ that $(aa') \ \varepsilon \in W((a'a)\varepsilon)$, and so there exists $x \in W(a'a) \cap < E_S >$ such that $x \ \varepsilon \ aa'$. Notice that $(cax)a(cac) = ca(xa'ax) = cax$, so that $cax \in W(a)$. It follows that

$$a(cax) \ \varepsilon \ aa'''(aa')$$

$$\varepsilon \ a(a''aa'''aa''aa'a)a'$$

$$\varepsilon \ aa''aa'''a'a \qquad (by \ (1))$$

$$\varepsilon \ a^2a'''a' = a^2c.$$

In a similar way, we get $(cax)a \ \varepsilon \ a'''a'aa = ca^2$. Therefore $a\rho a^2$, and so $K \subseteq \ker\rho$, which implies that $K = \ker\rho$, as required.

Up to now, Theorem 3.1 is a direct consequence of Lemmas 3.3, 3.4 and 3.5.

ACKNOWLEDGEMENTS: This research is supported by the Talent Youth's Foundation of Education Department of Anhui Province (2011SQRL040).

REFERENCES

1. Preston GB; Inverse semigroups. J. London. Math. Soc., 1954; 29: 396-403.
2. Pastijn F, Petrich M; Congruences on regular semigroups. Trans. Am. Math. Soc., 1986; 295: 607-633.
3. Gomes GMS; R-uniotent congruences on regular semigroups. Semigroup Forum, 1985; 31: 265-280.
4. Gomes GMS; Orthodox congruences on regular semigroup. Semigroup Forum, 1988; 37: 149-166.
5. Tan X; Rectangular group congruences on regular semigroups. Pure and Applied Mathematics, 2002; 18:161-164.
6. Luo YF; R-uniotent congruences on an eventually regular semigroup. Algebra colloquium, 2007; 14(1): 37-52.
7. LuoYF; Orthodox congruences on an eventually regular semigroup. Semigroup Forum, 2007; 74(2): 319-336.
8. Howie JM; Fundamentals of Semigrops Theory, Oxford: Clarendon Press, 1995.
9. Auinger K, Hall TE; Representations of semigroups by transformations and the congruence lattice of an eventually regular semigroup, Int. J. of Algebra and Computation, 1996; 6(6): 665-685.
10. Weipoltshammer B; Certain congruences on E-inverse E-semigroups. Semigroup Forum, 2002; 65: 233-248.