Scholars Journal of Physics, Mathematics and Statistics OPEN ACCESS
Abbreviated Key Title: Sch J Phys Math Stat
ISSN 2393-8056 (Print) | ISSN 2393-8064 (Online)
Journal homepage: https://saspublishers.com

Some Results on R-KKM Mappings and R-KKM Selections in GFC-Space

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DOI: 10.36347/sjpms.2022.v09i04.004 | **Received:** 11.04.2022 | **Accepted:** 19.05.2022 | **Published:** 24.05.2022

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Abstract **Review Article**

R-KKM mapping and R-KKM selection are introduced in GFC-space, some non-empty intersection theorems are proved, and some related results of Verma in G-H-space is generalized.
Keywords: GFC-space; R-KKM mapping; R-KKM selection; non-empty intersection theorem.

1. INTRODUCTION

Since 1987, after Horvath[1] gave H-space without linear structure by replacing convex hull with contractible set, Park and Kim[2] introduced G-convex spaces, Verma[3-8] introduced G-H-space and gave R-KKM selection in G-H-space, Ben-El-Mechaiekh[9] *et al*. introduced L-convex spaces, Ding[10] introduced FC-space without any convexity structure, Khanh[11, 12] *et al.* introduced GFC-space, and in this paper, we give R-KKM selection and R-KKM theorem in GFC-space and generalize some related results in recent literature.

2. PRELIMINARIES

Let X be a nonempty set. We denote by 2^X and $\langle X\rangle$ the family of all subsets of X and the family of all nonempty finite subsets of X. Let Δ_n be the standard (n-1) simplex with vertices $\{e_1,e_2,...,e_n\}$ in R^n. For any nonempty subset J of $\{1,2,...,n\}$, let Δ_J=$(\{e_j : j\in J\})$.

Definition 2.1. *([12])*

Let X be a topologic space, Y be a nonempty set, and Φ be a family of continuous mappings $\varphi:\Delta_n\to X$, $n\in N$. Then a triple (X,Y,Φ) is said to be a generalized finitely continuous topological space (GFC-space in short) if for each finite subset $N=\{y_1,y_2,...,y_n\}\in\langle Y\rangle$, there is $\varphi_N:\Delta_n\to X$ of the family Φ.

Definition 2.2.

Let (X,Y,Φ) be a GFC-space and $T:Y\to 2^X$ a multivalued mapping. T is said to be a R-KKM mapping if for any $\{y_1,y_2,...,y_n\}\in\langle Y\rangle$ there exists a subset $\{x_1,x_2,...,x_n\}\in\langle X\rangle$ such that

$$\varphi_N(\Delta_k)\subset\cup_{j=1}^{k}T(y_{i_j})$$

for a subsimplex $(e_{i_1},e_{i_2},...,e_{i_k})$ of $(e_1,e_2,...,e_n)=\Delta_n$ for $\{i_1,i_2,...,i_k\}\subset\{1,2,...,n\}$.

Definition 2.3.

Let (X,Y,Φ) be a GFC-space, $x_1,x_2,...,x_n$ be n elements of X and, $M_1,M_2,...,M_n$ subsets of X. Elements $F=\{x_1,x_2,...,x_n\}\in\langle X\rangle$ are called a relative R-KKM selection for $M_1,M_2,...,M_n$ if for $\{i_1,i_2,...,i_k\}\subset\{1,2,...,n\}$, we have

$$\varphi_N(\Delta_k)\subset\cup_{j=1}^{k}M_{i_j}$$

Where $(e_{i_1},e_{i_2},...,e_{i_k})$ is a standard (n-1) subsimplex of $(e_1,e_2,...,e_n)$ in R^n.

Definition 2.4.

Let A be a subset of X . A is said to be compactly open (or compactly closed) in X if for each nonempty compact subset K of X , $D\cap K$ is open (or closed) in K .

Let X and Y be two sets and $S:X\to 2^Y$ be a set-valued mapping, then $S^{-1}:Y\to 2^X$ and $S^*:Y\to 2^X$ are defined as $S^{-1}(y)=\{x\in X: y\in S(x)\}$ and $S^*(y)=X\setminus S^{-1}(y)$, respectively. Obviously, $x\in S^*(y)$ when and only when $y\notin S(x)$.

3. R-KKM Type Theorems

Theorem3.1. Let (X,Y,Φ) be a GFC-space, and $M_1,M_2,...,M_n$ compact closed subsets of X . Suppose that $F=\{x_1,x_2,...,x_n\}\in\langle X\rangle$ is a R-KKM selection for $M_1,M_2,...,M_n$. Then we have $\bigcap_{i=1}^n M_i\neq\varnothing$.

Proof. Suppose $\bigcap_{i=1}^n M_i=\varnothing$, then we have $\varphi_N(\Delta_n)\subset X\setminus\bigcap_{i=1}^n M_i=\bigcup_{i=1}^n (X\setminus M_i)$. It follows that

$$\varphi_N(\Delta_n)=\bigcup_{i=1}^n ((X\setminus M_i)\cap\varphi_N(\Delta_n))$$

Since M_i is a compact closed subset, $\{(X\setminus M_i)\cap\varphi_N(\Delta_n)\}_{i=1}^n$ is a open cover of $\varphi_N(\Delta_n)$. Let $\{\psi_i\}_{i=1}^n$ be the continuous partition of unity subordinate to the open covering, then we have that for each $i\in\{1,2,...,n\}$ and $y\in\varphi_N(\Delta_n)$,

$$\psi_i(y)\neq 0\Leftrightarrow y\in (X\setminus M_i)\cap\varphi_N(\Delta_n) \tag{1}$$

Define a mapping $\Psi:\varphi_N(\Delta_n)\to\Delta_n$ by $\Psi(y)=\sum_{i=1}^n\psi_i(y)e_i$, $\forall y\in\varphi_N(\Delta_n)$ .Obviously, $\Psi\circ\varphi_N:\Delta_n\to\Delta_n$ is continuous. By the Brouwer fixed-point theorem, there exists a point $z_0\in\Delta_n$ such that $z_0=\Psi\circ\varphi_N(z_0)$. Let $u_0=\varphi_N(z_0)$, then we have

$$u_0=\varphi_N(z_0)=\varphi_N\circ\Psi\circ\varphi_N(z_0)=\varphi_N\circ\Psi(u_0)$$

and

$$\Psi(u_0)=\sum_{i=1}^n\psi_i(u_0)e_i=\sum_{j\in\Delta_{J(u_0)}}\psi_j(u_0)e_j\in\Delta_{J(u_0)}$$

where $J(u_0)=\{j\in\{1,2,...,n\}:\psi_j(u_0)\neq 0\}$ and $\Delta_{J(u_0)}=co\{e_j: j\in J(u_0)\}$.

From equation (1), we know that, $u_0\in(X\setminus M_j)\cap\varphi_N(\Delta_n), \forall j\in J(u_0)$, then we have

$$u_0\notin M_j,\ \forall j\in J(u_0) \tag{2}$$

Since $F=\{x_1,x_2,...,x_n\}\in\langle X\rangle$ is a R-KKM selection for $M_1,M_2,...,M_n$, therefore, we have $\varphi_N(\Delta_{J(u_0)})\subset\bigcup_{j\in J(u_0)}M_j$. Then we have $u_0=\varphi_N(z_0)\in\varphi_N(\Delta_{J(u_0)})\subset\bigcup_{j\in J(u_0)}M_j$.

Thus, there exists j_0 such that $u_0\in M_{j_0}$ which contradicts the equation (2). Therefore, we have $\bigcap_{i=1}^n M_i\neq\varnothing$.

Remark 3.1. Theorem 3.1 generalizes Theorem 2.1 of [3] from G-H-space to GFC-space and from closed subsets to compact closed subsets.

Theorem3.2. Let (X,Y,Φ) be a GFC-space, and $M_1,M_2,...,M_n$ compact open subsets of X . Suppose that $F=\{x_1,x_2,...,x_n\}\in\langle X\rangle$ is a R-KKM selection for $M_1,M_2,...,M_n$. Then we have $\bigcap_{i=1}^n M_i\neq\varnothing$.

Proof. Suppose $\bigcap_{i=1}^n M_i=\varnothing$, then we have $\varphi_N(\Delta_n)\cap(\bigcap_{i=1}^n M_i)=\varnothing$. It follows that $\varphi_N(\Delta_n)=\bigcup_{i=1}^n(\varphi_N(\Delta_n)\setminus(M_i\cap\varphi_N(\Delta_n)))$

Since $\varphi_N(\Delta_n)$ is compact in Y, M_i is a compact open subset in X , then for each $i\in\{1,2,...,n\}$, $\varphi_N(\Delta_n)\cap M_i$ is open in $\varphi_N(\Delta_n)$.

For each $z\in\Delta_n$, let $I(z)=\{i\in\{1,2,...,n\}:\varphi_N(z)\notin M_i\}$ and $S(z)=co(\{e_i: i\in I(z)\})$. If for some $z\in\Delta_n$, $I(z)=\varnothing$. Then we have $\varphi_N(z)\in M_i$ for all

$i \in \{1,2,...,n\}$ which contradicts the assumption $\varphi_N(\Delta_n) \cap (\bigcap_{i=1}^n M_i) = \varnothing$.

Therefore we can assume that $I(z) \neq \varnothing$ for each $z \in \Delta_n$ and hence $S(z)$ is a nonempty compact convex subset of Δ_n for each $z \in \Delta_n$. Since $\bigcup_{i \notin I(z)} (\varphi_N(\Delta_n) \setminus (M_i \cap \varphi_N(\Delta_n)))$ is closed in $\varphi_N(\Delta_n)$, we have that $U = \Delta_n \setminus \varphi_n^{-1}(\bigcup_{i \notin I(z)} (\varphi_N(\Delta_n) \setminus (M_i \cap \varphi_N(\Delta_n))))$ is an open neighborhood of z in Δ_n. For each $z' \in U$, we have $\varphi_N(z') \subset M_i$ for all $i \notin I(z)$ and hence $I(z') \subset I(z)$. It follows that $S(z') \subset S(z)$ for all $z' \in U$.

This shows that $S : \Delta_n \to 2^{\Delta_n}$ is an upper semicontinuous set-valued mapping with nonempty compact convex values. By the Kakutani fixed point theorem, there exists a $z_0 \in \Delta_n$ such that $z_0 \in S(z_0)$. Note that $F = \{x_1, x_2, ..., x_n\} \in \langle X \rangle$ is a R-KKM selection for $M_1, M_2, ..., M_n$, then we have $\varphi_N(z_0) \in \varphi_N(S(z_0)) \subset \bigcup_{i \in I(z_0)} M_i$. Hence there exists a $i_0 \in I(z_0)$ such that $\varphi_N(z_0) \in M_{i_0}$. By the definition of $I(z_0)$, we have $\varphi_N(z_0) \notin M_i$ for each $i \in I(z_0)$, which is a contradiction. Therefore $\bigcap_{i=1}^n M_i \neq \varnothing$.

Remark 3.2. Theorem 3.2 proves that Theorem 3.1 also holds under the condition of compactly opening.

Theorem3.3. Let (X,Y,Φ) be a GFC-space, $T : Y \to 2^X \setminus \{\varnothing\}$ be a set-valued mapping with compactly open values. If T is a R-KKM mapping, then $\bigcap_{y \in Y} T(y) \neq \varnothing$.

Proof. Since $T : Y \to 2^X \setminus \{\varnothing\}$ be a set-valued mapping with compactly open values, then for each $\{y_1, y_2, ..., y_n\} \in \langle Y \rangle$, there exists $\{x_1, x_2, ..., x_n\} \in \langle X \rangle$ such that for each $\{i_1, i_2, ..., i_k\} \subset \{1,2,...,n\}$, we have $\varphi_N(\Delta_k) \subset \bigcup_{j=1}^k T(y_{i_j})$. It is also known that $T(y)$ is compactly closed, then by Theorem 2.1 we have $\bigcap_{y \in Y} T(y) \neq \varnothing$.

Remark 3.3. Theorem 3.3 generalizes Theorem 2.2 in [3] from G-H-space to GFC-space and from closed-valued mapping to compactly closed-valued mapping by removing the assumption of compactness of the space (X,Y,Φ).

Theorem3.4. Let (X,Y,Φ) be a GFC-space and $S, T : X \to 2^X$ two multivalued mappings such that:

(i) Tx is closed and $Sx \subset Tx$ for all $x \in X$;

(ii) $x \in Sx$ for all $x \in X$;

Then $\bigcap_{x \in X} T(x) \neq \varnothing$.

Proof. By Theorem 3.3, it is only necessary to prove that $T : X \to 2^X$ is a R-KKM mapping. Suppose $T : X \to 2^X$ is not a R-KKM mapping, there exists $\{x_1, x_2, ..., x_n\} \in \langle X \rangle$ such that $\varphi_N(\Delta_k) \not\subset \bigcup_{j=1}^k T(x_{i_j})$. There exists $u \in \varphi_N(\Delta_k)$ such that for each $j \in \{1,2,...,k\}$, we have $u \notin T(x_{i_j})$. Therefore $x_{i_j} \in T^*(u)$. Since $Sx \subset Tx$ for all $x \in X$, then we have $T^*(u) \subset S^*(u), \forall x \in X$. Therefore $x_{i_j} \in T^*(u) \subset S^*(u)$, $\forall j \in \{1,2,...,k\}$. Then we have $u \notin S(x_{i_j})$ for all $j \in \{1,2,...,k\}$ which contradicts condition (ii). Therefore $\bigcap_{x \in X} T(x) \neq \varnothing$.

Remark3.4. Theorem 3.4 generalizes Theorem 2.3 in [3] from G-H-space to GFC-space and from closed-valued mappings to compactly closed-valued mappings by removing the compactness assumption of the space (X,Y,Φ) and the convexity assumption of S^*x.

Theorem3.5. Let (X,Y,Φ) be a GFC-space and $S, T : X \to 2^X$ two multivalued mappings such that:

(i) $Sx \subset Tx$ for all $x \in X$;

(ii) $S^{-1}y$ is a compactly open subset in X;

Then there exists $x_0 \in X$ such that $x_0 \in Tx_0$.

Proof. Assume that the conclusion is not valid, then we have $x \notin Tx$ for each $x \in X$. Therefore

$x \in T^* x$, $\forall x \in X$. From condition (ii), it follows that $S^* y$ is a compactly open subset in X. Since $Sx \subset Tx$, we have $T^* y \subset S^* y$ for each $y \in X$. From Theorem 3.4, we have $\bigcap_{x \in X} S^* x \neq \varnothing$. Let $u \in \bigcap_{x \in X} S^* x$, then $u \in S^* x$. Therefore $x \notin S(u)$, $\forall x \in X$, then we have $S(u) = \varnothing$. This contradicts the definition of mapping S. Therefore, there exists $x_0 \in X$ such that $x_0 \in Tx_0$.

Remark 3.5. Theorem 3.5 generalizes Theorem 2.4 in [3] from G-H-space to GFC-space and from open-valuedness of the inverse map to compactly open-valuedness of the inverse map, removing the assumption of compactness of the space (X, Y, Φ) and the assumption of convexity of Tx.

REFERENCE参考文献

1. Horvath, C. (1987). Some results on multivalued mappings and inequalities without convexity[J]. *Pure and Appl. Math. Series*, 106: 99-106.
2. Park, S., & Kim, H. (1997). Foundations of the KKM theory on generalized convex spaces. *Journal of Mathematical Analysis and Applications*, *209*(2), 551-571.
3. Verma, R. U. (1999). Some results on R-KKM mappings and R-KKM selections and their applications. *Journal of Mathematical Analysis and Applications*, *232*(2), 428-433.
4. Verma, R. U. (1999). GH-KKM type theorems and their applications to a new class of minimax inequalities. *Computers & Mathematics with Applications*, *37*(8), 45-48.
5. Verma, R. U. (1999). Generalized GH-convexity and minimax theorems in GH-spaces. *Computers & Mathematics with Applications*, *38*(1), 13-18.
6. Verma, R. U. (1999). Relative KKM type selection theorems and their applications. *Applied mathematics letters*, *12*(7), 39-43.
7. Verma, R. U. (1999). On a generalized class of minimax inequalities. *Journal of mathematical analysis and applications*, *240*(2), 361-366.
8. Verma, R. U. (1999). Role of generalized KKM type selections in a class of minimax inequalities. *Applied mathematics letters*, *12*(4), 71-74.
9. Ben-El-Mechaiekh, H., Chebbi, S., Florenzano, M., & Llinares, J. V. (1998). Abstract convexity and fixed points. *Journal of Mathematical Analysis and Applications*, *222*(1), 138-150.
10. Ding, X. P. (2005). Maximal element theorems in product FC-spaces and generalized games. *Journal of Mathematical Analysis and Applications*, *305*(1), 29-42.
11. Hai, N. X., Khanh, P. Q., & Quan, N. H. (2009). Some existence theorems in nonlinear analysis for mappings on GFC-spaces and applications. *Nonlinear Analysis: Theory, Methods & Applications*, *71*(12), 6170-6181.
12. Khanh, P. Q., Quan, N. H., & Yao, J. C. (2009). Generalized KKM-type theorems in GFC-spaces and applications. *Nonlinear Analysis: Theory, Methods & Applications*, *71*(3-4), 1227-1234.
13. 方敏, 丁协平. (2003). L-凸空间内的广义 LR-KKM 型定理及应用[J]. 四川师范大学学报: 自然科学版, *26*(5): 461-463.
14. Ding, X. P., Liou, Y. C., & Yao, J. C. (2005). Generalized R-KKM type theorems in topological spaces with applications. *Applied mathematics letters*, *18*(12), 1345-1350.
15. Ding, X. P. (2002). Generalized L-KKM type theorems in L-convex spaces with applications. *Computers & Mathematics with Applications*, *43*(10-11), 1249-1256.
16. 邓方平, & 丁协平. (2005). 拓扑空间中的 KKM 选择与 KKM 定理. *四川师范大学学报: 自然科学版*, *28*(4), 402-404.