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Concrete Uninorms on Bounded Lattices

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Abstract **Review Article**

This paper presents a concrete method to construct uninorms via closure operators and interior operators on an arbitrary bounded lattices, where some sufficient and necessary conditions on the underlying t-norms and t-conorms are required. Finally, we illustrate how our new construction method is different from some existing methods for the constructions on bounded lattices.

Keywords: Bounded lattices; closure operator; interior operator; uninorms.

1. INTRODUCTION

Uninorms on the unit interval $[0,1]$ were introduced by Yager and Rybalov [1]. The uninorms as a generalization of t-norms and t-conorms [2] were applied to various fields, such as fuzzy logic, fuzzy set theory, expert systems, neural networks and so on [3-5].

Due to the fact that the bounded lattices [6] case in more general, uninorms [7-23] on the bounded lattices were defined and extensively studied. Uninorms on an arbitrary bounded lattice were first proposed by Karaçal and Mesiar [7]. Particularly, they constructed the weakest and the strongest uninorms. Then the new methods for constructing uninorms were obtained by Çaylı *et al.*, [14, 15, 17]. Subsequently, some methods to construct uninorms via closure (interior) operators on some bounded lattices were first proposed by Ouyang and Zhang [18]. Then, some other methods to construct uninorms via t-subnorms (t-subconorms) on some appropriate bounded lattices L with a neutral element $e \in L \setminus \{0,1\}$ were first introduced by Ji [21].

Uninorms on bounded lattices are conjunctive or disjunctive. In this paper, we introduce a new method which changes the disjunctive and conjunctive properties of uninorms on L for constructing uninorms based on a t-norm T_e on $[0,e]$ and t-conorm S_e on $[e,1]$ under some additional constraints. Our method is different from some existing methods for the constructions on bounded lattices. By concretizing Theorem3.1, we can get Theorem 3 and Theorem 4 in [18].

The rest of this paper is organized as follows. Section 2, we recall some preliminaries. Section 3, we introduce a new method for constructing uninorms on bounded lattices. Finally, some conclusions are made in Section 4.

2. Preliminaries

In this following, we recall some basic notions and results related to lattices and aggregation functions on bounded lattices.

Definition 2.1([6]) A lattice $(L,\le)$ is bounded if it has top and bottom elements, which are written as 1 and 0, respectively, that is, there exit two elements $1,0 \in L$ such that $0 \le x \le 1$ for all $x \in L$.

Throughout this article, unless stated otherwise, we denote L as a bounded lattice with the top and bottom elements 1 and 0, respectively.

Definition 2.2([6]) Let L be a bounded lattice, $a,b \in L$ with $a \le b$. A subinterval $[a,b]$ of L is defined as

$$[a,b]=\{x \in L : a \le x \le b\}. \quad \text{..........} (1)$$

Similarly, we can define

$$[a,b[=\{x \in L : a \le x < b\},$$

$]a,b] = \{x \in L : a < x \leq b\}$ and $]a,b[= \{x \in L : a < x < b\}$. If a and b are incomparable, then we use the notation $a \parallel b$. For $e \in L \setminus \{0,1\}$, we denote the set of all incomparable elements with e by I_e, that is, $I_e = \{x \in L \mid x \parallel e\}$.

Definition 2.3([2]) Let $(L, \leq, 0, 1)$ be a bounded lattice.

(i) An operation $T : L^2 \to L$ is called a t-norm on L if it is commutative, associative, and increasing with respect to both variables, and it has the neutral element $1 \in L$, that is, $T(1, x) = x$ for all $x \in L$.

(ii) An operation $S : L^2 \to L$ is called a t-conorm on L if it is commutative, associative, and increasing with respect to both variables, and it has the neutral element $0 \in L$, that is, $S(0, x) = x$ for all $x \in L$.

Definition 2.4 ([7]) Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $U : L^2 \to L$ is called a uninorm on L (a uninorm if L is fixed) if it is commutative, associative, and increasing with respect to both variables, and it has the neutral element $e \in L$, that is, $U(e, x) = x$ for all $x \in L$.

Proposition 2.1 ([7]) Let $(L, \leq, 0, 1)$ be a bounded lattice and U be a uninorm on L with neutral element $e \in L \setminus \{0,1\}$. Then we have the following:

(i) $T_e = U|[0,e]^2 : [0,e]^2 \to [0,e]$ is a t-norm on $[0,e]$.

(ii) $S_e = U|[e,1]^2 : [e,1]^2 \to [e,1]$ is a t-conorm on $[e,1]$.

T_e and S_e given in proposition 2.1 are called the underlying t-norm and t-conorm of a uninorm U on a bounded lattice L with the neutral element e, respectively. Throughout this study, we denote T_e as the underlying t-norm and S_e as the underlying t-conorm of a given uninorm U on L.

Definition 2.5 ([2]) Let L be a lattice.

(i)A mapping $cl : L \to L$ is called a closure operator on L if, for all $x, y \in L$, it satisfies the following three conditions:

(1) $x \leq cl(x)$;

(2) $cl(x \vee y) = cl(x) \vee cl(y)$;

(3) $cl(cl(x)) = cl(x)$.

(ii)A mapping $\operatorname{int} : L \to L$ is called an interior operator on L if, for all $x, y \in L$, it satisfies the following three conditions:

(1) $\operatorname{int}(x) \leq x$;

(2) $\operatorname{int}(x \wedge y) = \operatorname{int}(x) \wedge \operatorname{int}(y)$;

(3) $\operatorname{int}(\operatorname{int}(x)) = \operatorname{int}(x)$.

Theorem 2.1([23]) Let $(L, \leq, 0, 1)$ be a bounded lattice with $e \in L \setminus \{0,1\}$.

(i)If T_e is a t-norm on $[0,e]$ and R is a t-subconorm on L, then the function $U_{R,0}(x,y) : L^2 \to L$ defined by

$$U_{R,0}(x,y) = \begin{cases} T_e(x,y) & (x,y) \in [0,e]^2 \\ x \vee y & (x,y) \in]0,e] \times]e,1] \cup]e,1] \times]0,e] \\ y & (x,y) \in]0,e] \times I_e \\ x & (x,y) \in I_e \times]0,e] \\ 0 & (x,y) \in \{0\} \times I_e \cup I_e \times \{0\} \cup \{0\} \times]e,1] \cup]e,1] \times \{0\} \\ R(x,y) & otherwise, \end{cases} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

is a uninorm on L with the neutral element $e \in L \setminus \{0,1\}$ iff $T_e(x,y) > 0$ for all $x, y > 0$.

(ii)If S_e is a t-conorm on $[e,1]$ and F is a t-subnorm on L, then the function $U_{F,1}(x,y) : L^2 \to L$ defined by

$$U_{F,1}(x,y)=\begin{cases} S_e(x,y) & (x,y)\in[e,1]^2 \\ x\wedge y & (x,y)\in[0,e[\times[e,1[\cup[e,1[\times[0,e[\\ y & (x,y)\in[e,1[\times I_e \\ x & (x,y)\in I_e\times[e,1[\\ 1 & (x,y)\in\{1\}\times[0,e[\cup[0,e[\times\{1\}\cup\{1\}\times I_e\cup I_e\times\{1\} \\ F(x,y) & otherwise, \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

is a uninorm on L with the neutral element $e\in L\setminus\{0,1\}$ iff $S_e(x,y)<1$ for all $x,y<1$.

Theorem 2.2([18]) Let $(L,\leq,0,1)$ be a bounded lattice with $e\in L\setminus\{0,1\}$.

(i)If T_e is a t-norm on $[0,e]$ and cl is a closure operator on L, then the function $U_{cl}(x,y):L^2\to L$ is a uninorm on L with the neutral element e, where

$$U_{cl}(x,y)=\begin{cases} T_e(x,y) & (x,y)\in[0,e]^2 \\ x\vee y & (x,y)\in[0,e]\times]e,1]\cup]e,1]\times[0,e] \\ y & (x,y)\in[0,e]\times I_e \\ x & (x,y)\in I_e\times[0,e] \\ cl(x)\vee cl(y) & otherwise. \end{cases} \qquad \ldots\ldots\ldots\ldots (4)$$

(ii)If S_e is a t-conorm on $[e,1]$ and int is an interior operator on L, then the function $U_{\text{int}}(x,y):L^2\to L$ is a uninorm on L with the neutral element e, where

$$U_{\text{int}}(x,y)=\begin{cases} S_e(x,y) & (x,y)\in[e,1]^2 \\ x\wedge y & (x,y)\in[0,e[\times[e,1]\cup[e,1]\times]0,e] \\ y & (x,y)\in[e,1]\times I_e \\ x & (x,y)\in I_e\times[e,1] \\ \text{int}(x)\wedge\text{int}(y) & otherwise. \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots (5)$$

Theorem 2.3([12]) Let $(L,\leq,0,1)$ be a bounded lattice with $e\in L\setminus\{0,1\}$.

(i)If S_e is a t-conorm on $[e,1]$ such that $S_e(x,y)<1$ for all $x,y<1$, then the function $U_1:L^2\to L$ is a uninorm on L with the neutral element $e\in L\setminus\{0,1\}$, where

$$U_1(x,y)=\begin{cases} S_e(x,y) & (x,y)\in[e,1]^2 \\ x & (x,y)\in I_e\times[e,1[\\ y & (x,y)\in[e,1[\times I_e \\ x\vee y & (x,y)\in\{1\}\times[0,e]\cup[0,e]\times\{1\}\cup\{1\}\times I_e\cup I_e\times\{1\} \\ x\wedge y & otherwise. \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots (6)$$

(ii)If T_e is a t-norm on $[0,e]$ such that $T_e(x,y)>0$ for all $x,y>0$, then the function $U_2(x,y):L^2\to L$ is a uninorm on L with the neutral element $e\in L\setminus\{0,1\}$, where

$$U_2(x,y)=\begin{cases} T_e(x,y) & (x,y)\in[0,e]^2 \\ x & (x,y)\in I_e\times]0,e] \\ y & (x,y)\in]0,e]\times I_e \\ x\wedge y & (x,y)\in\{0\}\times I_e\cup I_e\times\{0\}\cup\{0\}\times[e,1]\cup[e,1]\times\{0\} \\ x\vee y & otherwise. \end{cases} \quad \text{………….. (7)}$$

3. New methods to construct concrete uninorms on bounded lattices

In this section, we introduce a new method which changes the disjunctive and conjunctive properties of uninorms on L for constructing uninorms on an arbitrary bounded lattices with a neutral element $e\in L\setminus\{0,1\}$. Our results can be used to enrich the classes of uninorms on bounded lattices.

Theorem 3.1 Let $(L,\le,0,1)$ be a bounded lattice with $e\in L\setminus\{0,1\}$.

(i)If T_e is a t-norm on $[0,e]$ and cl is a closure operator on L, then the function $U_{cl,0}(x,y):L^2\to L$ defined by

$$U_{cl,0}(x,y)=\begin{cases} T_e(x,y) & (x,y)\in[0,e]^2 \\ x\vee y & (x,y)\in]0,e]\times]e,1]\cup]e,1]\times]0,e] \\ y & (x,y)\in]0,e]\times I_e \\ x & (x,y)\in I_e\times]0,e] \\ 0 & (x,y)\in\{0\}\times I_e\cup I_e\times\{0\}\cup\{0\}\times]e,1]\cup]e,1]\times\{0\} \\ cl(x)\vee cl(y) & otherwise, \end{cases} \quad \text{……… (8)}$$

is a uninorm on L with the neutral element $e\in L\setminus\{0,1\}$ iff $T_e(x,y)>0$ for all $x,y>0$.

(ii)If S_e is a t-conorm on $[e,1]$ and int is an interior operator on L, then the function $U_{\text{int},1}(x,y):L^2\to L$ defined by

$$U_{\text{int},1}(x,y)=\begin{cases} S_e(x,y) & (x,y)\in[e,1]^2 \\ x\wedge y & (x,y)\in[0,e[\times[e,1[\cup[e,1[\times[0,e[\\ y & (x,y)\in[e,1[\times I_e \\ x & (x,y)\in I_e\times[e,1[\\ 1 & (x,y)\in\{1\}\times[0,e[\cup[0,e[\times\{1\}\cup\{1\}\times I_e\cup I_e\times\{1\} \\ \text{int}(x)\wedge\text{int}(y) & otherwise, \end{cases} \quad \text{…………… (9)}$$

is a uninorm on L with the neutral element $e\in L\setminus\{0,1\}$ iff $S_e(x,y)<1$ for all $x,y<1$.

Proof. We give the proof of the fact that $U_{cl,0}$ is a uninorm iff $T_e(x,y)>0$ for all $x,y>0$. The same result for $U_{\text{int},1}$ can be obtained using similar arguments.

Necessity. Let the function $U_{cl,0}$ be a uninorm on L with the neutral element $e\in L\setminus\{0,1\}$. We prove that $T_e(x,y)>0$ for all $x,y>0$. Assume that there are some elements $x\in]0,e[$ and $y\in]0,e[$ such that $T_e(x,y)=0$. If $z\in I_e$, then we obtain $U_{cl,0}(x,U_{cl,0}(y,z))=U_{cl,0}(x,z)=z$ and $U_{cl,0}(U_{cl,0}(x,y),z)=U_{cl,0}(T_e(x,y),z)=0$. Since $T_e(x,y)=0$, the associativity property is violated. Then $U_{cl,0}$ is not a uninorm on L which is a contradiction. Hence, $T_e(x,y)>0$ for all $x,y>0$.

Sufficiency. $R(x,y)=cl(x)\vee cl(y)$. Observe that R is a t-subconorm on L. Thus, we obtain that $U_{cl,0}$ is a uninorm on L with a neutral element $e \in L \setminus \{0,1\}$ by Theorem 2.1.

It is worth pointing out that the bounded conditions in Theorem 2.3 are sufficient and necessary.

Theorem 3.2 Let $(L, \leq, 0, 1)$ be a bounded lattice with $e \in L \setminus \{0,1\}$.

(i)If T_e is a t-norm on $[0,e]$, then the function $U_{cl,0}(x,y): L^2 \to L$ defined by

$$U_1(x,y) = \begin{cases} S_e(x,y) & (x,y) \in [e,1]^2 \\ x & (x,y) \in I_e \times [e,1[\\ y & (x,y) \in [e,1[\times I_e \\ x \vee y & (x,y) \in \{1\} \times [0,e] \cup [0,e] \times \{1\} \cup \{1\} \times I_e \cup I_e \times \{1\} \\ x \wedge y & \textit{otherwise}, \end{cases} \tag{10}$$

is a uninorm on L with the neutral element $e \in L \setminus \{0,1\}$ iff $T_e(x,y) > 0$ for all $x, y > 0$.

(ii)If S_e is a t-conorm on $[e,1]$, then the function $U_{\mathrm{int},1}(x,y): L^2 \to L$ defined by

$$U_2(x,y) = \begin{cases} T_e(x,y) & (x,y) \in [0,e]^2 \\ x & (x,y) \in I_e \times]0,e] \\ y & (x,y) \in]0,e] \times I_e \\ x \wedge y & (x,y) \in \{0\} \times I_e \cup I_e \times \{0\} \cup \{0\} \times [e,1] \cup [e,1] \times \{0\} \\ x \vee y & \textit{otherwise}, \end{cases} \tag{11}$$

is a uninorm on L with the neutral element $e \in L \setminus \{0,1\}$ iff $S_e(x,y) < 1$ for all $x, y < 1$.

Proof. We give the proof of the fact that U_1 is a uninorm iff $T_e(x,y) > 0$ for all $x, y > 0$. The same result for U_2 can be obtained using similar arguments.

Necessity. Let the function U_1 be a uninorm on L with the neutral element $e \in L \setminus \{0,1\}$. We prove that $T_e(x,y) > 0$ for all $x, y > 0$. Assume that there are some elements $x \in]0,e[$ and $y \in]0,e[$ such that $T_e(x,y) = 0$. If $z \in I_e$, then we obtain $U_1(x, U_1(y,z)) = U_1(x,z) = z$ and $U_1(U_1(x,y),z) = U_1(T_e(x,y),z) = 0$. Since $T_e(x,y) = 0$, the associativity property is violated. Then U_1 is not a uninorm on L which is a contradiction. Hence, $T_e(x,y) > 0$ for all $x, y > 0$.

Sufficiency. The result can be proved in a manner similar to the proof of Theorem 2.3.

Corollary 3.1 Let $(L, \leq, 0, 1)$ be a bounded lattice with $e \in L \setminus \{0,1\}$ and $cl(x) = x$ in Theorem 3.1, then $U_{cl,0}$ in Theorem 3.1 is equal to U_1 in Theorem 3.2.

Corollary 3.2 Let $(L, \leq, 0, 1)$ be a bounded lattice with $e \in L \setminus \{0,1\}$ and $\mathrm{int}(x) = x$ in Theorem 3.1, then $U_{\mathrm{int},1}$ in Theorem 3.1 is equal to U_2 in Theorem 3.2.

4. CONCLUSION

In this article, we investigate the construction of uninorms on arbitrary bounded lattices with $e \in L \setminus \{0,1\}$, where some sufficient and necessary conditions on the underlying t-norms and t-conorms are required. Then we investigate the relation between introduced methods and some other approaches. By concretizing Theorem 3.1, we can get Theorem 3.2. In the future, we well continue to construct new uninorms.

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