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New Concrete Constructions of Nullnorms on Bounded Lattices

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Abstract **Review Article**

The structure of the nullnorms are the basis for the study of nullnorms. This paper presents two concrete methods to construct nullnorms via triangular subconorms (triangular subnorms) and triangular norms (triangular conorms) on bounded lattices, then gets two constructions of nullnorms on bounded lattices via triangular subconorms (triangular subnorms) and triangular norms (triangular conorms).

Keywords: nullnorms; bounded lattices; triangular subconorms; triangular norms.

1. INTRODUCTION

The concept of nullnorm on unit interval $[0,1]$ was introduced by Calvo [1]. From a theoretical point of view, nullnorm is important. Meanwhile, it is also widely used in many fields, such as expert systems, fuzzy quantifiers, neural networks, fuzzy logic [2].

Since bounded lattices [3] are more general than unit intervals [2-9], most studies of nullnorms focus on bounded lattices [10-12]. Based on the existence of t-norms and t-conorms on bounded lattices, Karaçal et al. [10] defined nullnorms on bounded lattices and proposed three construction methods of nullnorms on bounded lattices with an arbitrary zero element $a \in L \setminus \{0,1\}$. Later, some construction methods of nullnorms on bounded lattices were also proposed by Ertuğr *et al.*, [11, 19, 20]. For the first time, Xie, Ji [18] constructed nullnorms via triangular subconorms (triangular subnorms) on bounded lattices.

In order to complete the structure of nullnorms on bounded lattices, two concrete methods to construct nullnorms via triangular subconorms (triangular subnorms) and triangular norms (triangular conorms) on bounded lattices are presented in this paper.

2. Preliminaries

In this section, we will recall some basic definitions and theorems which will be applied to this paper.

Definition 2.1.[13] A lattice $(L,\leq)$ is bounded if it has top and bottom elements, which are written as 1 and 0, respectively; that is, two elements $0,1 \in L$ exist such that $0 \leq x \leq 1$ for all $x \in L$.

Throughout this paper, unless stated otherwise, we denote L as a bounded lattice with the top and bottom elements 1 and 0, respectively.

Definition 2.2.[13] Given a bounded lattice $(L,\leq,0,1)$ and $a,b \in L$, $a \leq b$, a subset $[a,b]$ of L is defined as $[a,b] = \{x \in L \mid a \leq x \leq b\}$. Similarly, denote $[a,b) = \{x \in L \mid a \leq x < b\}$, $(a,b] = \{x \in L \mid a < x \leq b\}$ and $(a,b) = \{x \in L \mid a < x < b\}$. If a and b are incomparable, we use the notation $a \parallel b$. The set of all elements which are incomparable with a are denoted by I_a.

Definition 2.3.[14] Let $(L,\leq,0,1)$ be a bounded lattice.

(1) An operation $T: L^2 \to L$ is called a triangular norm (t-norm for short) if it is commutative, associative, increasing with respect to both variables and has the neutral element $1 \in L$ such that $T(x,1) = x$ for all $x \in L$.

(2) An operation $S: L^2 \to L$ is called a triangular conorm (t-conorm for short) if it is commutative, associative, increasing with respect to both variables and has the neutral element $0 \in L$ such that $S(x,0) = x$ for all $x \in L$.

Definition 2.4.[15] Let $(L, \le, 0, 1)$ be a bounded lattice. A commutative, associative, non-decreasing in each variable function $V: L^2 \to L$ is called a nullnorm if an element $a \in L$ exists such that $V(x,0) = x$ for all $x \le a$ and $V(x,1) = x$ for all $x \ge a$.

It is easy to see that $V(x,a) = a$ for all $x \in L$, thus a is the zero element for V.

Theorem 2.1.[16] Let $(L, \le, 0, 1)$ be a bounded lattice and $V: L^2 \to L$ be a nullnorm on L with the zero element a. Then,

(1) $V_{[0,a]^2} : [0,a]^2 \to [0,a]$ is a t-conorm on $[0,a]$;

(2) $V_{[a,1]^2} : [a,1]^2 \to [a,1]$ is a t-norm on $[a,1]$.

Definition 2.5.[14] Let $(L, \le, 0, 1)$ be a bounded lattice.

(1) An operation $F: L^2 \to L$ is called a t-subnorm on L if it is commutative, associative, increasing with respect to both variables and $F(x,y) \le x \wedge y$ for all $x, y \in L$.

(2) An operation $R: L^2 \to L$ is called a t-subconorm on L if it is commutative, associative, increasing with respect to both variables and $R(x,y) \ge x \vee y$ for all $x, y \in L$.

Theorem 2.2.[17] Let $(L, \le, 0, 1)$ be a bounded lattice, $a \in L \setminus \{0,1\}$, S is a t-conorm on $[0,a]$, and T is a t-norm on $[a,1]$. Then, the functions $V_T^S, V_S^T : L^2 \to L$ can be defined as:

$$V_T^S = \begin{cases} S(x,y) & (x,y) \in [0,a]^2 \\ T(x,y) & (x,y) \in [a,1]^2 \\ S(x \wedge a, y \wedge a) & (x,y) \in [0,a] \times I_a \cup I_a \times [0,a] \cup I_a \times I_a \\ a & \text{otherwise}, \end{cases}$$

$$V_S^T = \begin{cases} S(x,y) & (x,y) \in [0,a]^2 \\ T(x,y) & (x,y) \in [a,1]^2 \\ T(x \wedge a, y \wedge a) & (x,y) \in [a,1] \times I_a \cup I_a \times [a,1] \cup I_a \times I_a \\ a & \text{otherwise}, \end{cases}$$

And they are nullnorms on L with zero element a.

In order to reduce the complexity in the proof of associativity, we introduce the following theorem.

Theorem 2.3.[21] Let S be a nonempty set and A, B, C, D be subsets of S. Let H be a commutative binary operation on S. Then H is associative on $A \cup B \cup C \cup D$ both of the following statements hold:

(1) $H(H(x,y),z) = H(x, H(y,z))$ for all $(x,y,z) \in (A,A,A) \cup (B,B,B) \cup (C,C,C)$ $\cup (D,D,D) \cup (A,A,B) \cup (A,B,B) \cup (A,A,C) \cup (A,C,C,) \cup (A,A,D) \cup (A,D,D) \cup$ $(B,B,C) \cup (B,C,C) \cup (B,B,D) \cup (B,D,D) \cup (C,C,D) \cup (C,D,D)$.

(2) $H(H(x,y),z) = H(x, H(y,z)) = H(H(x,z),y)$ for all $(x,y,z) \in (A,B,C) \cup$ $(A,B,D) \cup (A,C,D) \cup (B,C,D)$.

3. New Constructions of Nullnorms on Bounded Lattices

In this section, we will recall some basic definitions and theorems which will be applied to this paper.

Theorem 3.1. Let $(L, \leq, 0, 1)$ be a bounded lattice, $a \in L \setminus \{0,1\}$, R is a t-subconorm on $[0,a]$, and T is a t-norm on $[a,1]$.Then, the function $V_T^R : L^2 \to L$ can be defined as:

$$V_T^R = \begin{cases} R(x,y) & (x,y) \in (0,a]^2 \\ x \vee y & (x,y) \in \{0\} \times [0,a] \cup [0,a] \times \{0\} \\ T(x,y) & (x,y) \in [a,1]^2 \\ (x \wedge a) \vee (y \wedge a) & (x,y) \in [0,a] \times I_a \cup I_a \times [0,a] \cup I_a \times I_a \\ \text{a} & \text{otherwise,} \end{cases}$$

And it is nullnorm on L with zero element a, if and only if "$x \wedge a = 0$ for all $x \in I_a$".

Proof. Sufficiency: The commutativity of V_T^R can be proven directly based on its description. Similarly, we can express $V_T^R(x,0) = x$ for all $x \in [0,a]$ and $V_T^R(x,1) = x$ for all $x \in [a,1]$. Now, we only need to proof monotonicity and associativity.

Monotonicity: Let us prove that if $x \leq y$, then $V_T^R(x,z) \leq V_T^R(y,z)$ for all $z \in L$.

1. It is obvious that $V_T^R(x,z) \leq V_T^R(y,z)$, if $x = 0$.
2. $x \in (0,a]$

2.1. $y = (0,a]$

2.1.1. $z = 0$

$V_T^R(x,z) = x \leq y = V_T^R(y,z)$

2.1.2. $z \in (0,a]$

$V_T^R(x,z) = R(x,z) \leq R(y,z) = V_T^R(y,z)$

2.1.3. $z \in [a,1]$

$V_T^R(x,z) = a = V_T^R(y,z)$

2.1.4. $z \in I_a$

$V_T^R(x,z) = (x \wedge a) \vee (z \wedge a) = x \leq y = (y \wedge a) \vee (z \wedge a) = V_T^R(y,z)$

2.2. $y \in [a,1]$

2.2.1. $z = 0$

$V_T^R(x,z) = x \leq a = V_T^R(y,z)$

2.2.2. $z \in (0,a]$

$V_T^R(x,z) = R(x,z) \leq a = V_T^R(y,z)$

2.2.3. $z \in [a,1]$

$V_T^R(x,z) = a \leq T(y,z) = V_T^R(y,z)$

2.2.4. $z \in I_a$

$V_T^R(x,z) = (x \wedge a) \vee (z \wedge a) = x \leq a = V_T^R(y,z)$

3. $x \in [a,1]$

3.1. $y \in [a,1]$

3.1.1. $z=0$

$V_T^R(x,z)=a=V_T^R(y,z)$

3.1.2. $z\in(0,a]$

$V_T^R(x,z)=a=V_T^R(y,z)$

3.1.3. $z\in[a,1]$

$V_T^R(x,z)=T(x,z)\le T(y,z)=V_T^R(y,z)$

3.1.4. $z\in I_a$

$V_T^R(x,z)=a=V_T^R(y,z)$

4. $x\in I_a$

4.1. $y\in I_a$

4.1.1. $z=0$

$V_T^R(x,z)=(x\wedge a)\vee(z\wedge a)=0=(y\wedge a)\vee(z\wedge a)=V_T^R(y,z)$

4.1.2. $z\in(0,a]$

$V_T^R(x,z)=(x\wedge a)\vee(z\wedge a)=z=(y\wedge a)\vee(z\wedge a)=V_T^R(y,z)$

4.1.3. $z\in[a,1]$

$V_T^R(x,z)=a=V_T^R(y,z)$

4.1.4. $z\in I_a$

$V_T^R(x,z)=(x\wedge a)\vee(z\wedge a)=0=(y\wedge a)\vee(z\wedge a)=V_T^R(y,z)$

4.2. $y\in[a,1]$

4.2.1. $z=0$

$V_T^R(x,z)=(x\wedge a)\vee(z\wedge a)=0\le a=V_T^R(y,z)$

4.2.2. $z\in(0,a]$

$V_T^R(x,z)=(x\wedge a)\vee(z\wedge a)=z\le a=V_T^R(y,z)$

4.2.3. $z\in[a,1]$

$V_T^R(x,z)=a\le T(y,z)=V_T^R(y,z)$

4.2.4. $z\in I_a$

$V_T^R(x,z)=(x\wedge a)\vee(z\wedge a)=0\le a=V_T^R(y,z)$

Associativity: It can be shown that $V_T^R\left(V_T^R(x,y),z\right)=V_T^R\left(x,V_T^R(y,z)\right)$ for all $x,y,z\in L$. By Theorem 2.3, We only need to consider the following cases:

1. $x=0$, $y=0$, $z=0$

$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(0,z)=0=V_T^R(x,0)=V_T^R\left(x,V_T^R(y,z)\right)$$

2. $x\in(0,a]$, $y\in(0,a]$, $z\in(0,a]$

$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(R(x,y),z)=R(R(x,y),z)$$
$$=R(x,R(y,z))=V_T^R(x,R(y,z))=V_T^R\left(x,V_T^R(y,z)\right)$$

3. $x\in[a,1]$, $y\in[a,1]$, $z\in[a,1]$

$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(T(x,y),z)=T(T(x,y),z)$$
$$=T(x,T(y,z))=V_T^R(x,T(y,z))=V_T^R\left(x,V_T^R(y,z)\right)$$

4. $x\in I_a$, $y\in I_a$, $z\in I_a$
$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(0,z)=0=V_T^R(x,0)=V_T^R\left(x,V_T^R(y,z)\right)$$

5. $x=0$, $y=0$, $z\in(0,a]$
$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(0,z)=z=V_T^R(x,z)=V_T^R\left(x,V_T^R(y,z)\right)$$

6. $x=0$, $y\in(0,a]$, $z\in(0,a]$
$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(y,z)=R(y,z)=V_T^R(x,R(y,z))=V_T^R\left(x,V_T^R(y,z)\right)$$

7. $x=0$, $y=0$, $z\in[a,1]$
$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(0,z)=a=V_T^R(x,a)=V_T^R\left(x,V_T^R(y,z)\right)$$

8. $x=0$, $y\in[a,1]$, $z\in[a,1]$
$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(a,z)=a=V_T^R(x,T(y,z))=V_T^R\left(x,V_T^R(y,z)\right)$$

9. $x=0$, $y=0$, $z\in I_a$
$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(0,z)=0=V_T^R(x,0)=V_T^R\left(x,V_T^R(y,z)\right)$$

10. $x=0$, $y\in I_a$, $z\in I_a$
$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(x,z)=0=V_T^R(x,0)=V_T^R\left(x,V_T^R(y,z)\right)$$

11. $x\in(0,a]$, $y\in(0,a]$, $z\in[a,1]$
$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(R(x,y),z)=a=V_T^R(x,a)=V_T^R\left(x,V_T^R(y,z)\right)$$

12. $x\in(0,a]$, $y\in[a,1]$, $z\in[a,1]$
$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(a,z)=a=V_T^R(x,T(y,z))=V_T^R\left(x,V_T^R(y,z)\right)$$

13. $x\in(0,a]$, $y\in(0,a]$, $z\in I_a$
$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(R(x,y),z)=R(x,y)=V_T^R(x,y)=V_T^R\left(x,V_T^R(y,z)\right)$$

14. $x\in(0,a]$, $y\in I_a$, $z\in I_a$
$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(x,z)=x=V_T^R(x,0)=V_T^R\left(x,V_T^R(y,z)\right)$$

15. $x\in[a,1]$, $y\in[a,1]$, $z\in I_a$
$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(T(x,y),z)=a=V_T^R(x,a)=V_T^R\left(x,V_T^R(y,z)\right)$$

16. $x\in[a,1]$, $y\in I_a$, $z\in I_a$
$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(a,z)=a=V_T^R(x,0)=V_T^R\left(x,V_T^R(y,z)\right)$$

17. $x=0$, $y\in(0,a]$, $z\in[a,1]$
$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(y,z)=a=V_T^R(x,a)=V_T^R\left(x,V_T^R(y,z)\right)$$
$$=V_T^R(a,y)=V_T^R\left(V_T^R(x,z),y\right)$$

18. $x=0$, $y\in(0,a]$, $z\in I_a$

$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(y,z)=y=V_T^R(x,y)=V_T^R\left(x,V_T^R(y,z)\right)$$
$$=V_T^R(0,y)=V_T^R\left(V_T^R(x,z),y\right)$$

19. $x=0$, $y\in[a,1]$, $z\in I_a$

$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(a,z)=a=V_T^R(x,a)=V_T^R\left(x,V_T^R(y,z)\right)$$
$$=V_T^R(0,y)=V_T^R\left(V_T^R(x,z),y\right)$$

20. $x\in(0,a]$, $y\in[a,1]$, $z\in I_a$

$$V_T^R\left(V_T^R(x,y),z\right)=V_T^R(a,z)=a=V_T^R(x,a)=V_T^R\left(x,V_T^R(y,z)\right)$$
$$=V_T^R(x,y)=V_T^R\left(V_T^R(x,z),y\right)$$

Therefore, V_T^R is a nullnorm on L with the zero element a.

Necessity: Let V_T^R is a nullnorm on L with the zero element a and $x\wedge a\in(0,a)$ for all $x\in I_a$. Let $x\in(0,a)$, $y=0$, $z\in I_a$, $R(x,y)=x\vee y\vee a$, then we get $V_T^R\left(V_T^R(x,y),z\right)=V_T^R(x,z)=x\vee(z\wedge a)$ and $V_T^R\left(x,V_T^R(y,z)\right)=V_T^R(x,z\wedge a)=R(x,z\wedge a)=x\vee(z\wedge a)\vee a=a$. We know that $x\vee(z\wedge a)<a$, so $V_T^R\left(V_T^R(x,y),z\right)\neq V_T^R\left(x,V_T^R(y,z)\right)$. This is contradictory to the associativity of nullnorm. Therefore, it is must be $x\wedge a=0$ for all $x\in I_a$.

Theorem 3.2. Let $(L,\leq,0,1)$ be a bounded lattice, $a\in L\setminus\{0,1\}$, S is a t-conorm on $[0,a]$, and F is a t-subnorm on $[a,1]$.Then, the function $V_S^F:L^2\to L$ can be defined as:

$$V_S^F=\begin{cases} F(x,y) & (x,y)\in[a,1)^2 \\ x\wedge y & (x,y)\in\{1\}\times[a,1)\cup[a,1)\times\{1\} \\ S(x,y) & (x,y)\in[0,a]^2 \\ (x\vee a)\wedge(y\vee a) & (x,y)\in[a,1]\times I_a\cup I_a\times[a,1]\cup I_a\times I_a \\ \text{a} & \text{otherwise,} \end{cases}$$

And it is nullnorm on L with zero element a, if and only if "$x\vee a=1$ for all $x\in I_a$".

Proof. This proof is similar to the proof of theorem 3.1.

Remark 3.11 The biggest difference between the construction methods of nullnorm proposed in this paper and the construction methods of nullnorm proposed in theorem 2.2 is that: We replace the triangular conorm (triangular norm) with the triangular subconorm (triangular subnorm), and the most important thing is that we give the necessary and sufficient condition for those construction methods.

4. CONCLUSION

In previous studies, nullnorms on bounded lattices have been defined and studied extensively. Moreover, the concrete construction of nullnorm on bounded lattices is still an active research field.

In this paper, we consider the particularity of specific bounded lattices, and according to the concrete constructions of nullnorms form theorem 2.2, we present two concrete methods to construct nullnorms via triangular subconorms (triangular subnorms) and triangular norms (triangular conorms) on bounded lattices. In the following research, we will continue to find and use different aggregation operators to construct new nullnorms on bounded lattices, so as to make the structure of nullnorms on bounded lattices more complete.

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