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Bifurcation of Travelling Wave Solutions for (3+1)-dimensional mKdV-ZK Equation

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Abstract **Review Article**

Since the (3+1)-dimensional modified Korteweg de Vries- Zakharov–Kuznetsov (mKdV-ZK equation) was proposed, many methods have been used to find its exact solution, but some solutions are always missed in the calculation process. In this paper, the mKdV-ZK equation is studied by the planar dynamical systems method, the bifurcation diagram and phase diagram under different parameters are obtained, and the solution of the equation is numerically simulated. Finally, the traveling wave solutions of the mKdV-ZK equation is constructed under different parameters.
Keywords: mKdV-ZK equation bifurcation diagram phase diagtam Numerical Simulation travelling wave solutions.

1. INTRODUCTION

The study of exact traveling wave solutions of nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. For example, wave phenomena observed in fluid mechanics, plasmas, and elastic media are often modeled by bell-shaped and kink-shaped solitary wave solutions. In the past few decades, researchers have put a lot of effort into how to construct accurate solutions to nonlinear evolution equations, and have proposed and developed numerous methods for solving nonlinear evolution equations. These methods include Jacobi elliptic function method [1], Hirota bilinear transformation [2], Weierstrass function method [3], Darboux transformation and Backlund transformation [4], Wronskian skills [5], Homotopy perturbation method [6], function method [7], Symmetric method [8, 9], Homogeneous equilibrium method [8-11], tanh-coth expansion method [12-14], F-expansion [15], exp-function [16-19], The Painleve truncated expansion method [20].

In 1988, When K.P.D as and Frank Verheest study on plasma, which consisting of any number of adiabatic positive and negative ion species in addition to the presence of isothermal electrons, the derived the famous equation-(3+1)-dimensions KdV-ZK equation. Finally, the solitons of the mKdV-ZK equation are investigated by the small-K perturbation expansion method of Rowlands and Infeld, stability criteria and growth rates of instabilities are derived [21]. Since then, many people have studied the equation. For example, Guiqiong Xu presented an elliptic equation method, and the elliptic function solutions of rational forms for the (3+1)-dimensional are derived [22]. In 2012, Hasibun Naher *et al.*, study (3+1)-dimensional mKdV-ZK equation via the generalized and improved (G'/G)-expansion method, they find the solutions in hyperbolic, trigonometric and rational function form [23]. In 2021, Li Yan et al. used the rational sine-Gordan expansion method to obtain the solutions of some trigonometric functions, periodic functions, hyperbolic functions and rational functions of the (3+1)-dimensional mKdV-ZK equation, and introduced each solution at the same time. The corresponding physical meanings, and the corresponding numerical simulations are given [24]. In 2022, Raj Raj Kumar · Ravi Shankar Verma obtained a series of new similarity solutions of the (3+1)-dimensional mKdV-ZK equation by using the classical Lie symmetric analysis method [25].

Although many exact solutions of the (3 + 1) -dimensional mKdV-ZK equation have been obtained by the above method, some solutions may be lost. In this paper, we will use the plane dynamic system method to study the equation. The plane dynamic system method can clearly reveal how the solution of the equation evolves when the parameters of the equation change.

2. The bifurcation analysis for the (3+1)-dimensional mKdV-ZK equation

Consider the following nonlinear evolution equation:

$$u_t+\beta u^2u_x+u_{xxx}+u_{xyy}+u_{xzz}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Do the traveling wave transform of equation (1), $u(x,y,z)=u(\xi)$, $\xi=x+ay+bz-ct$, where $c\neq 0$ is the wave speed. From this it can be obtained

$$-cu'+\beta u^2u'+(1+a^2+b^2)u^{(3)}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

Where ' means $d/d\xi$. Integrate once over ξ in (2), the following results can be obtained

$$-cu+\frac{1}{3}\beta u^3+(1+a^2+b^2)u^{(2)}=e_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

Where e_1 is the integral constant. From (3) we can solve $u^{(2)}$

$$u^{(2)}=\frac{e_1}{1+a^2+b^2}+\frac{c}{1+a^2+b^2}u-\frac{\beta}{3(1+a^2+b^2)}u^3 \ldots\ldots\ldots \text{(4)}$$

Equation (4) is equivalent to the following plane system

$$\begin{cases} u'=y \\ y'=\dfrac{e_1}{1+a^2+b^2}+\dfrac{c}{1+a^2+b^2}u-\dfrac{\beta}{3(1+a^2+b^2)}u^3 \end{cases} \ldots\ldots\ldots \text{(5)}$$

Obviously, it is a planar Hamiltonian system with the first integra

$$H(u,y)=\frac{1}{2}y^2+\frac{\beta}{12(1+a^2+b^2)}u^4-\frac{c}{2(1+a^2+b^2)}u^2-\frac{e_1}{1+a^2+b^2}u \ldots\ldots\ldots\ldots \text{(6)}$$

We are interested in looking for the possible zeros of the expression

$$f(u)=\frac{e_1}{1+a^2+b^2}+\frac{c}{1+a^2+b^2}u-\frac{\beta}{3(1+a^2+b^2)}u^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(7)}$$

take the first derivative of u in (7) and let the result be 0, we have

$$f'(u)=\frac{c}{1+a^2+b^2}-\frac{\beta}{1+a^2+b^2}u^2=0 \ldots\ldots\ldots \text{(8)}$$

From (8) we have the following result

$$u_1=\sqrt{\frac{c}{\beta}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(9)}$$

$$u_2=-\sqrt{\frac{c}{\beta}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(10)}$$

Therefore $u_1>u_2$.

Bring u_1 and u_2 into function $f(u)$ respectively

$$r_1=f(u_1)=\frac{e_1}{1+a^2+b^2}+\frac{2c\sqrt{\dfrac{c}{\beta}}}{3(1+a^2+b^2)} \ldots\ldots \text{(11)}$$

$$r_2 = f(u_2) = \frac{e_1}{1+a^2+b^2} - \frac{2c\sqrt{\frac{c}{\beta}}}{3(1+a^2+b^2)} \quad \text{............ (12)}$$

There $c > 0$, for $c < 0$ with similar results, this case will be omitted here for convenience. At the same time, for the convenience of calculation, we set the value of β to 1, so that we have

$$\frac{2c\sqrt{\frac{c}{\beta}}}{3(1+a^2+b^2)} > 0 \quad \text{... (13)}$$

$$r_1 - r_2 = \frac{4c\sqrt{\frac{c}{\beta}}}{3(1+a^2+b^2)} > 0 \quad \text{............................ (14)}$$

We have $r_1 = f(u_1) > f(u_2) = r_2$.

2.1 For the equilibrium points and the number of system (5), the following conclusions are drawn:

(1) if $r_1 < 0$, there is only one equilibrium point for the system, denoted by $(u_1^*, 0)$ ($u_1^* < u_2 < u_1$), which is a center;

(2) if $r_2 > 0$, there is only one equilibrium point for the system, denoted by $(u_2^*, 0)$ ($u_2 < u_1 < u_2^*$), which is a center;

(3) if $r_1 = 0$, there are two equilibrium points for the system, denoted by $(u_{31}^*, 0)$, $(u_{32}^*, 0)$ ($u_{32}^* < u_2 < u_1 = u_{31}^*$), which $(u_{31}^*, 0)$ is a cusp, $(u_{32}^*, 0)$ is a center;

(4) if $r_2 = 0$, there are two equilibrium points for the system, denoted by $(u_{41}^*, 0)$, $(u_{42}^*, 0)$ ($u_{41}^* = u_2 < u_1 < u_{42}^*$), which $(u_{41}^*, 0)$ is a cusp, $(u_{42}^*, 0)$ is a center;

(5) if $r_1 > 0$ and $r_2 < 0$, there are three equilibrium points for the system, denoted by $(u_{51}^*, 0)$, $(u_{52}^*, 0)$, $(u_{53}^*, 0)$ ($u_{53}^* < u_2 < u_{52}^* < u_1 < u_{51}^*$), which $(u_{51}^*, 0)$ and $(u_{53}^*, 0)$ are center, and $(u_{52}^*, 0)$ is a saddle point.

Proof. According to the definitions of r_1 and r_2, we can easily conclude that function $f(u)$ is monotonically decreasing in interval $(-\infty, u_2)$ and interval $(u_1, +\infty)$, and monotonically increasing in interval (u_2, u_1). For case (1), which is $r_1 < 0, r_2 < 0$, $f(u) = 0$ have only one zero, marked as u_1^* ($u_1^* < u_2 < u_1$), because u_1^* is in the interval $(-\infty, u_2)$, in this interval, the function $f(u)$ is monotonically decreasing, that is, any given u in this interval has $f'(u) < 0$ constant, so $f'(u_1^*) < 0$. From the properties of the Hamilton system the only unique equilibrium is a center.

The proofs for the other cases are similar, and the proofs for the other cases are omitted here for brevity.

In fact, for the case (5), it can be further subdivided into three cases $-r_2 = r_1$, $-r_2 < r_1$, and $-r_2 > r_1$. For these three cases, the system only has a homoclinic loop, and there is no heteroclinic loop.

To derive the details of the bifurcation, we fix the parameter $\beta = 1$, and we can make a bifurcation graph, as shown in Figure 1.

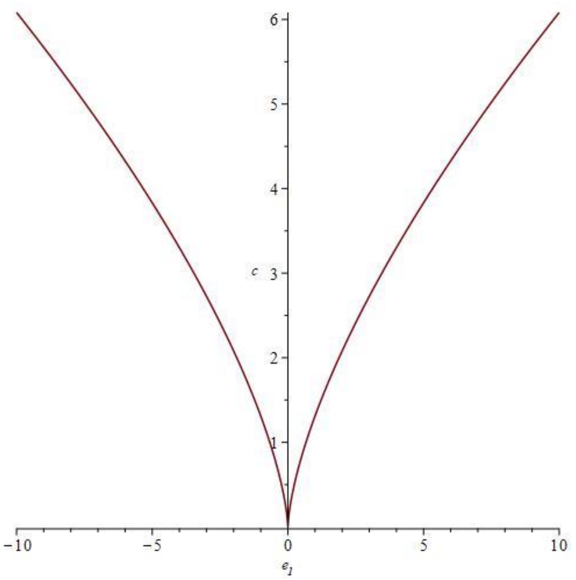

Figure 1: (e_1 -c) boundary diagram on the plane

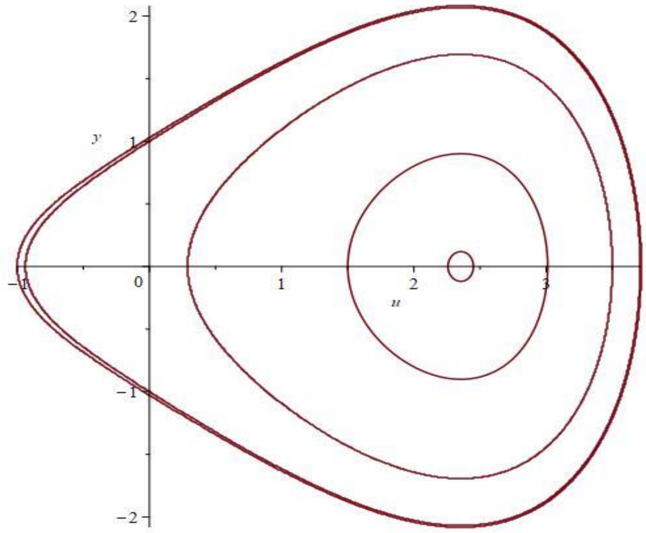

Figure 2: a=b=c=β=1, e_1 =2

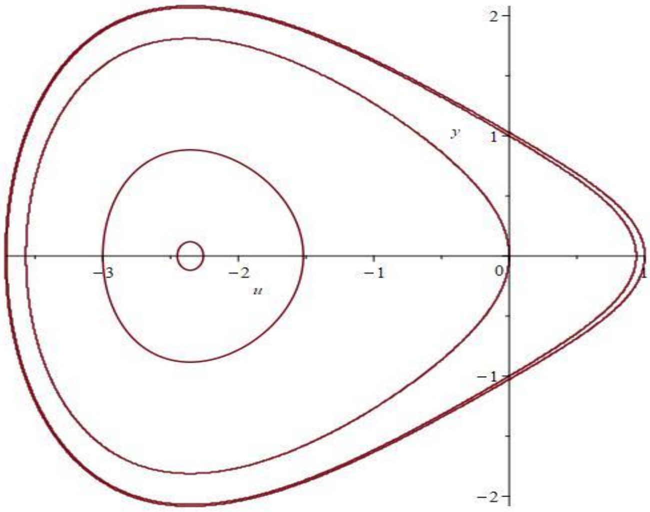

Figure 3: a=b=c=β=1, e_1 =-2

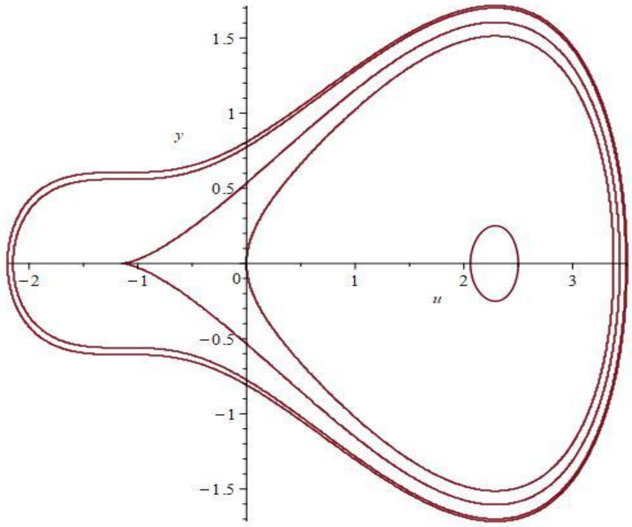

Figure 4: a=b=β=e_1 =1 c=$\sqrt[3]{\frac{9}{4}}$

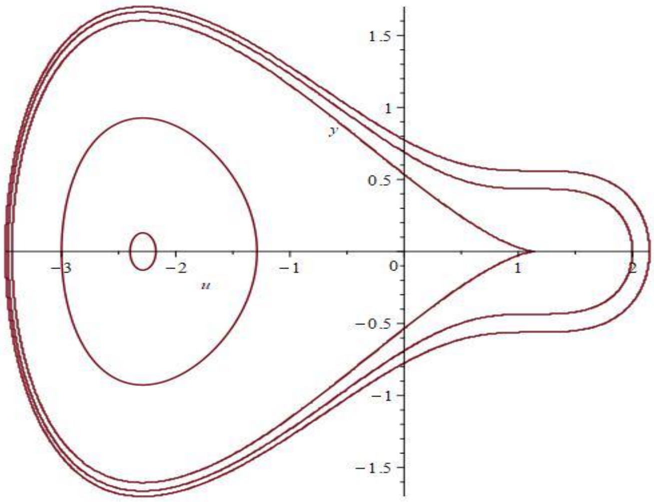

Figure 5: a=b=β=1 e_1 =-1 c=$\sqrt[3]{\frac{9}{4}}$

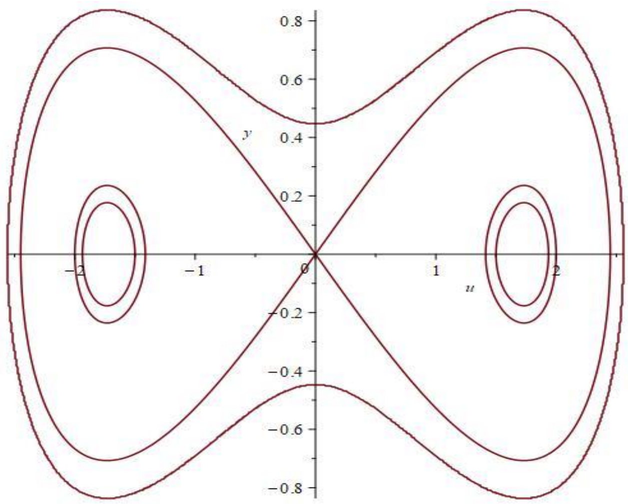

Figure 6: a=b=c=β=1 e_1 =0

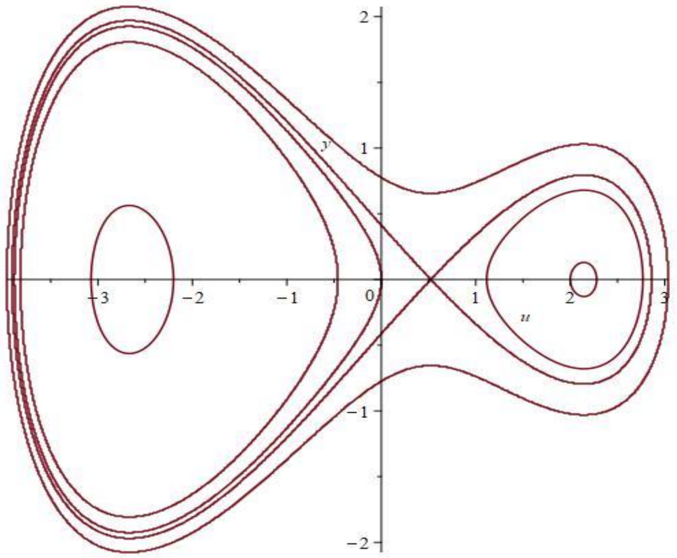

Figure 7: a=b=β=1 c=2 e_1 =-1

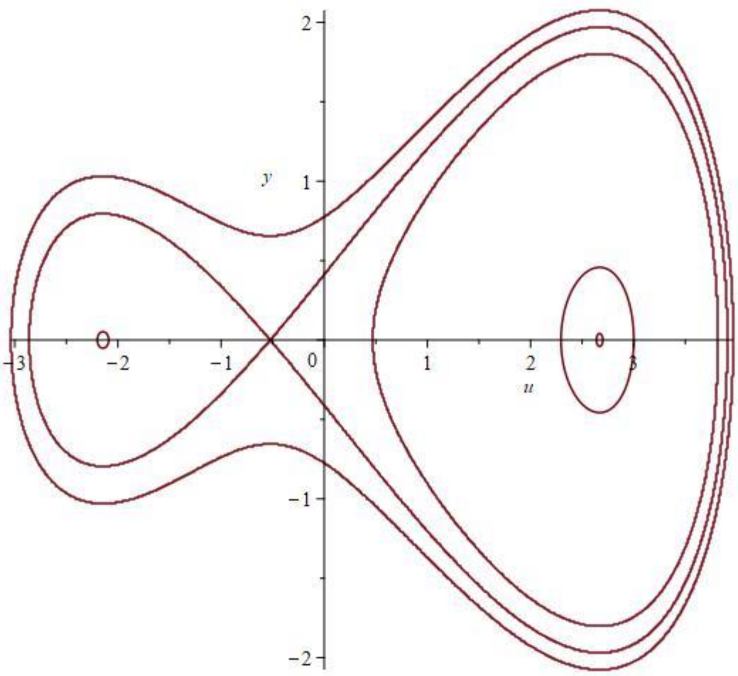

Figure 8: a=b=β= e_1 =1 c=2

3. Numerical simulations of bounded integral curve

We know that the bounded traveling wave $u(\xi)$ of system (1) corresponds to the bounded integral curve of equation (4), and equation (4) corresponds to the bounded orbit of (5). Therefore, we can know from the phase diagrams that each phase diagram contains bounded orbits.

3.1 Numerical simulation of periodic orbits

Taking Figure 2 as an example, we find that the equilibrium point is the center. Taking $u(0)=1$ as the initial value, the periodic integral curve of the equation can be simulated, as shown in Fig 9. Further, taking $u(0)=-1$, $u(0)=2$ and $u(0)=2.2$ as initial values respectively, we can simulate the degradation behavior of periodic integration, as shown in Figure 10-12.

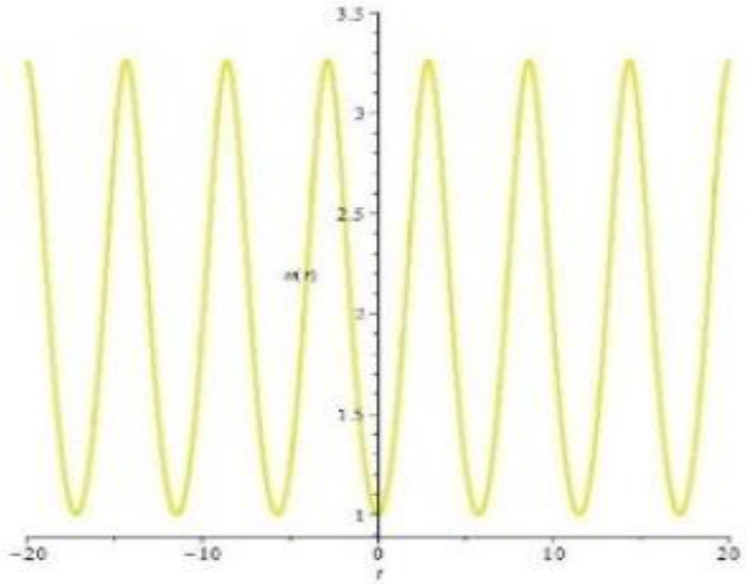

Figure 9: Numerical simulation of periodic orbits $u(0)=1, u^{'}(0)=0$

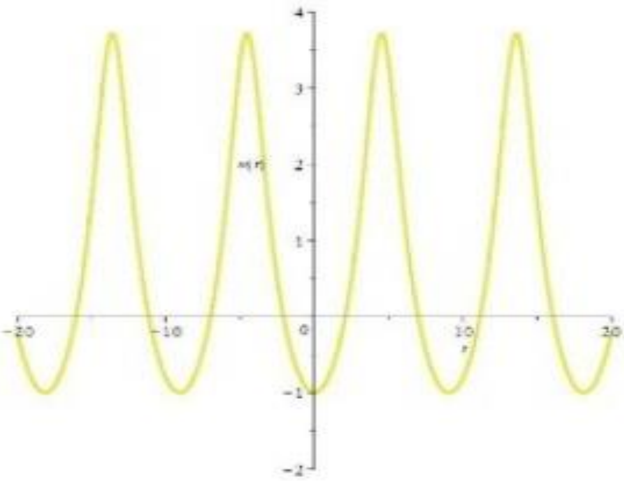

Figure 10: Numerical simulation of periodic orbits $u(0)=-1,\ u^{'}(0)=0$

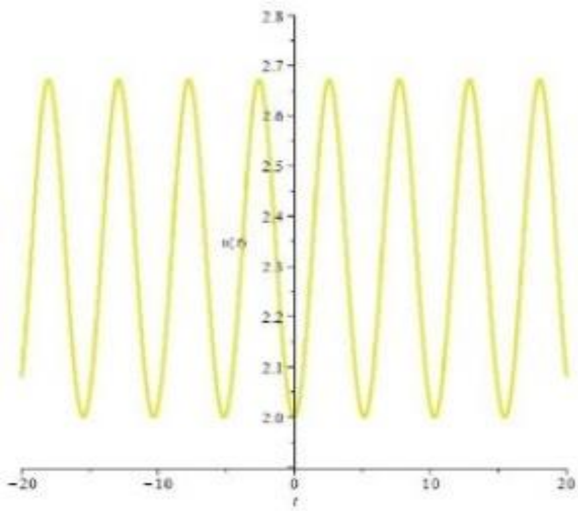

Figure 11: Numerical simulation of periodic orbits $u(0)=2, u^{'}(0)=0$

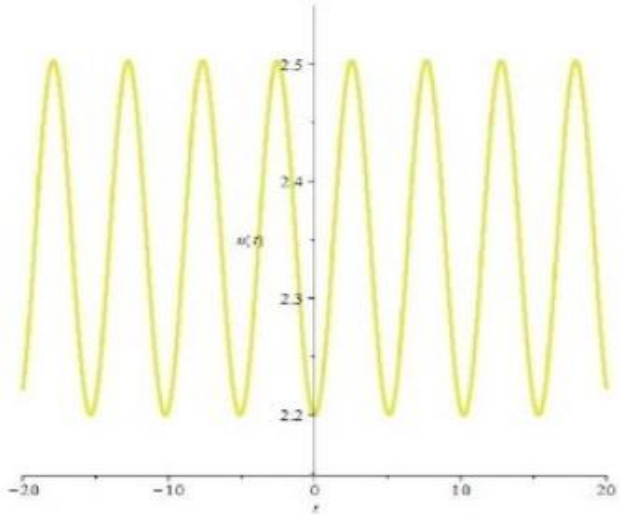

Figure 12: Numerical simulation of periodic orbits $u(0)=2.2\,, u^{'}(0)=0$

Looking at Fig 8-12, it can be found that when the initial value is close to the center, the periodic integral curve tends to be a triangular function periodic curve; when the initial value is far from the center, the periodic integral curve is a bell-shaped integral curve. Similar methods can be used to simulate the periodic integral curves corresponding to other closed orbits, and the results are similar, which will be omitted here for simplicity.

3.2 Numerical simulation of bounded integral curve corresponding to homoclinic orbit

Taking Fig 6 as an example, through calculation, it can be seen that $(\sqrt{3},0)$ and $(-\sqrt{3},0)$ are centers and $(0,0)$ is saddle point. By choosing $u(0)=\sqrt{6}$ and $u(0)=-\sqrt{6}$ as initial values respectively, we can simulate the bounded integral curve corresponding to the homoclinic orbit of system (1), as shown in Figure 13 & 14.

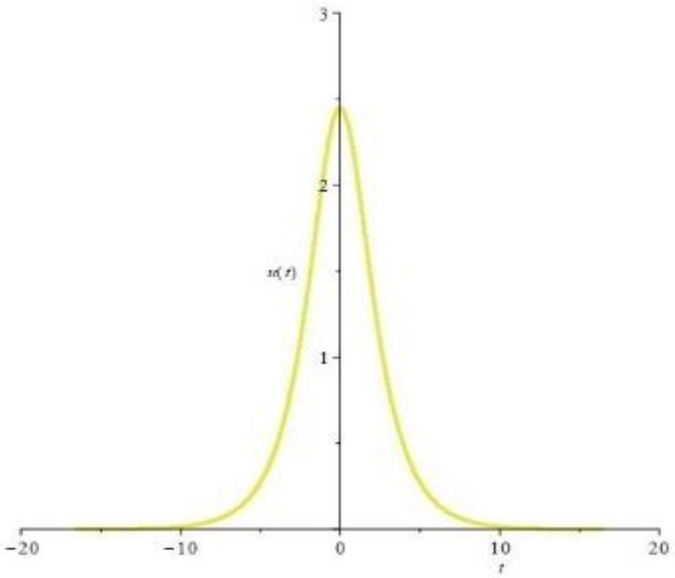

Figure 13: $u(0)=\sqrt{6}\,, u^{'}(0)=0$

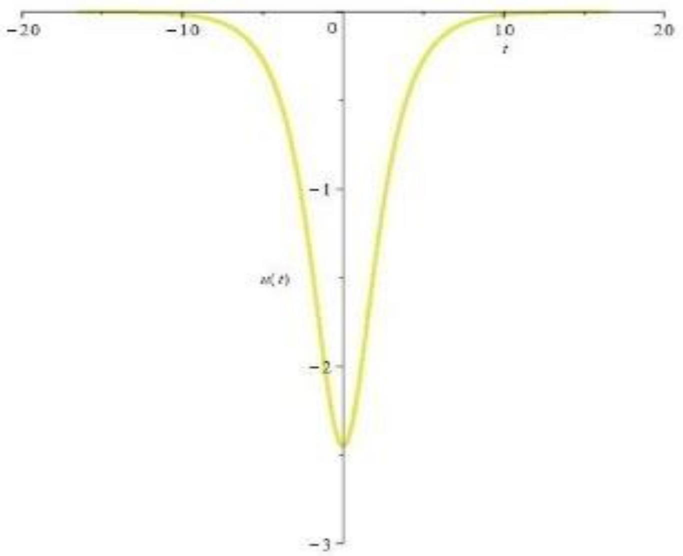

Figure 14: $u(0)=-\sqrt{6}\,,u^{'}(0)=0$

4. Explicit expressions of bounded integral curve

In this part, we will take Figure 6 as an example to calculate the exact expression of the bounded integral curve. Through observation, we can find that there are periodic orbits and homoclinic orbits in Fig 6. Furthermore, the periodic orbits can be divided into two cases: (1) the periodic orbits are located inside the homoclinic orbits; (2) The periodic orbit lies outside the homoclinic orbit.

4.1 The periodic solutions

Case 1:

from $H(u,y)=\frac{1}{2}y^2+\frac{\beta}{12(1+a^2+b^2)}u^4-\frac{c}{2(1+a^2+b^2)}u^2-\frac{e_1}{1+a^2+b^2}u$ and $u^{'}=y$, through calculation, it can be seen that:

$$y=\pm\sqrt{\frac{\beta}{6(1+a^2+b^2)}}\sqrt{-u^2+\frac{c}{\beta}u^2+\frac{12e_1}{\beta}u}\,.$$

By calculation, we have

$$y=\pm\sqrt{\frac{\beta}{6(1+a^2+b^2)}}\sqrt{(m-u)(u-n)(u-p)(u-q)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (15)$$

Where the parameters m, n, u, p, q all are reals and the relation $m>u>n>p>q$ holds, and select m as the initial value point, suppose the period of the closed orbit is 2T.

From (15) and $u^{'}=y$ we have

$$\int_u^m \frac{du}{\sqrt{\frac{\beta}{6(1+a^2+b^2)}}\sqrt{(m-u)(u-n)(u-p)(u-q)}}=\int_\xi^0 d\xi \quad \ldots\ldots\ldots\ldots\ldots (16)$$

Where $-T<\xi<0$,

$$-\int_m^u \frac{du}{\sqrt{\frac{\beta}{6(1+a^2+b^2)}}\sqrt{(m-u)(u-n)(u-p)(u-q)}}=\int_0^\xi d\xi \quad \ldots\ldots\ldots\ldots.. (17)$$

Where $0<\xi<T$.
From (16) and (17), the following results are obtained:

$$\int_u^m \frac{du}{\sqrt{\frac{\beta}{6(1+a^2+b^2)}}\sqrt{(m-u)(u-n)(u-p)(u-q)}}=|\xi| \qquad (18)$$

By looking up the table, we have the following results

$$\int_u^m \frac{du}{\sqrt{(m-u)(u-n)(u-p)(u-q)}}=gsn^{-1}(\sin\varphi,k)=gsn^{-1}(\sqrt{\frac{(n-q)(m-n)}{(m-n)(u-q)}},k)$$

$$\frac{1}{\sqrt{\frac{\beta}{6(1+a^2+b^2)}}}gsn^{-1}(\sqrt{\frac{(n-q)(m-u)}{(m-n)(u-q)}},k)=|\xi|$$

$$u=\frac{m(n-q)+q(m-n)sn^2(\frac{\sqrt{\frac{\beta}{6(1+a^2+b^2)}}}{g}|\xi|)}{n-q+(m-n)sn^2(\sqrt{\frac{\beta}{6(1+a^2+b^2)g^2}}|\xi|)}$$

Where $k^2=\frac{(m-n)(p-q)}{(m-p)(n-q)}$, $g=\frac{2}{\sqrt{(m-p)(n-q)}}$.

Case 2:

from $H(u,y)=\frac{1}{2}y^2+\frac{\beta}{12(1+a^2+b^2)}u^4-\frac{c}{2(1+a^2+b^2)}u^2-\frac{e_1}{1+a^2+b^2}u$ and $u'=y$, through calculation, it can be seen that:

$$y=\pm\sqrt{(m-u)(u-n)(u-c)(u-\bar{c})} \qquad (19)$$

Where c and $\bar{c}$ are conjugate complex numbers. The parameters m, n, u all are reals and the relation $m>u>n$, select m as the initial value point, and we suppose the period of the closed orbit is 2T, we have

$$\int_n^u \frac{du}{\sqrt{\frac{\beta}{6(1+a^2+b^2)}}\sqrt{(m-u)(u-n)(u-c)(u-\bar{c})}}=\int_0^\xi d\xi \qquad (20)$$

Where $0<\xi<T$,

$$-\int_u^n \frac{du}{\sqrt{\frac{\beta}{6(1+a^2+b^2)}}\sqrt{(m-u)(u-n)(u-c)(u-\bar{c})}}=\int_\xi^0 d\xi \qquad (21)$$

Where $-T<\xi<0$.
From (20) and (21)

$$\int_n^u \frac{du}{\sqrt{\frac{\beta}{6(1+a^2+b^2)}}\sqrt{(m-u)(u-n)(u-c)(u-\bar{c})}}=|\xi| \qquad (22)$$

The following results are obtained by looking up the table:

$$\int_n^u \frac{du}{\sqrt{(m-u)(u-n)(u-c)(u-\bar{c})}} = gcn^{-1}\left(\frac{(m-u)B-(u-n)A}{(m-u)B+(u-n)A}, k\right),$$

Where $b_1 = \dfrac{c+\bar{c}}{2}$, $a_1^2 = -\dfrac{(c-\bar{c})^2}{4}$, $A^2 = (m-b_1)^2 + a_1^2$, $B^2 = (n-b_1)^2 + a_1^2$, $g = \dfrac{1}{\sqrt{AB}}$, $k^2 = \dfrac{(m-n)^2-(A-B)^2}{4AB}$ .We can get the following results

$$u = n + \frac{m[B - Bcn(\dfrac{\sqrt{\frac{\beta}{6(1+a^2+b^2)}}|\xi|}{g})]}{A + Acn(\dfrac{\sqrt{\frac{\beta}{6(1+a^2+b^2)}}|\xi|}{g})}$$

4.2 Soliton solution

From $H(u,y) = \frac{1}{2}y^2 + \frac{\beta}{12(1+a^2+b^2)}u^4 - \frac{c}{2(1+a^2+b^2)}u^2 - \frac{e_1}{1+a^2+b^2}u$, we can write the expression of homoclinic loop

$$y = \pm\sqrt{\frac{\beta}{6(1+a^2+b^2)}} \sqrt{-u^4 + \frac{c}{\beta}u^2 + \frac{12e_1}{\beta}u} \quad \text{……………..…… (23)}$$

Further

$$y = \pm\sqrt{\frac{\beta}{6(1+a^2+b^2)}} \sqrt{(m-u)(u-n)^2(u-p)} \quad \text{………………. (24)}$$

Where the parameters m, n, u, p all are reals and the relation $m > u > n > p$ holds.

From (24) and $u' = y$, we have

$$\int_u^m \frac{du}{\sqrt{\frac{\beta}{6(1+a^2+b^2)}}\sqrt{(m-u)(u-n)^2(u-p)}} = \int_\xi^0 d\xi \quad \text{………... (25)}$$

Where $\xi < 0$,

$$-\int_m^u \frac{du}{\sqrt{\frac{\beta}{6(1+a^2+b^2)}}\sqrt{(m-u)(u-n)^2(u-p)}} = \int_0^\xi d\xi \quad \text{………. (26)}$$

Where $\xi > 0$.

So, from (25) and (26), we have

$$\int_n^m \frac{du}{\sqrt{\frac{\beta}{6(1+a^2+b^2)}}(u-n)\sqrt{(m-u)(u-p)}} = |\xi|, \int_m^n \frac{du}{(u-n)\sqrt{(m-u)(u-p)}} = -\sqrt{\frac{\beta}{6(1+a^2+b^2)}}|\xi|$$
……………….. (27)

Let $u-n=u^*$, so that $u=u^*+n$, we have

$$\int_m^u \frac{du}{(u-n)\sqrt{(m-u)(u-p)}} = \int_{m-n}^{u^*} \frac{du^*}{u^*\sqrt{-u^*+k_1u^*+k_2}} = -\sqrt{\frac{\beta}{6(1+a^2+b^2)}}|\xi| \text{………….. (28)}$$

Where $k_1=(m+p-2n)$, $k_2=(n-p)(m-n)$. Through calculation, it can be seen that

$$\int_{m-n}^{u^*} \frac{du^*}{u^*\sqrt{-u^*+k_1u^*+k_2}} = \frac{-\ln(\frac{2k_2+k_1u^*+2\sqrt{k_2}\sqrt{k_1u^*-(u^*)^2+k_2}}{u^*})}{\sqrt{k_2}} + c_1 \text{ ………….. (29)}$$

Where $c_1 = \dfrac{\ln(\frac{2k_2+k_1(m-n)+2\sqrt{k_2}\sqrt{k_1(m-n)-(m-n)^2+k_2}}{m-n})}{\sqrt{k_2}}$.

So, we finally have the following results

$$u^* = \frac{4k_2e^{\sqrt{k_2}(-\sqrt{\frac{\beta}{6(1+a^2+b^2)}}|\xi|-c_1)}}{(e^{\sqrt{k_2}(-\sqrt{\frac{\beta}{6(1+a^2+b^2)}}|\xi|-c_1)}-k_1)^2+4k_2}.$$

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