Research Article

On The Integral Solutions Of The Binary Quadratic Equation \( x^2 = 15y^2 - 11^t \)

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Abstract: The binary quadratic Diophantine equation represented by \( x^2 = 15y^2 - 11^t \), \( t \) odd is analysed for its non-zero distinct integer solutions. Employing the lemma of Brahmagupta, infinitely many integral solutions of the above Pell equation are obtained. The recurrence relations on the solutions are also presented. A few interesting relations among the solutions are given. Further, there exist no integer solutions when \( t \) is even.

Keywords: Binary quadratic, Pell equation., Integer solutions

INTRODUCTION

It is well known that the Pell equation \( x^2 - Dy^2 = \pm 1 \), \( (D > 0 \text{ and square free}) \) has always positive integer solutions. When \( N \neq 1 \), the Pell equation \( x^2 - Dy^2 = N \) may not have any positive integer solutions. For example the equations \( x^2 = 3y^2 - 1 \) and \( x^2 = 7y^2 - 4 \) have no positive integer solutions. When \( k \) is a positive integer and \( D \in \{ k^2 \pm 4, k^2 \pm 1 \} \), positive integer solutions of the equations \( x^2 - Dy^2 = \pm 4 \) and \( x^2 - Dy^2 = \pm 1 \) have been investigated by Jones in [4]. The same or similar equations are investigated in [3,6,9,10]. In [1,2,5,7,8,11,12,13] some specific Pell equation and their integer solutions are considered. In [14], the integer solutions of Pell equation \( x^2 - (k^2 + k)y^2 = 2^t \) has been considered. In [15], the Pell equation \( x^2 - (k^2 - k)y^2 = 2^t \) is analyzed for the integer solutions.

This communication concerns with the Pell equation \( x^2 = 15y^2 - 11^t \) and infinitely many positive integer solutions are obtained when \( t \) is odd. The recurrence relations on the solutions are also given. Further, it is observed that, when \( t \) is even there exist no integer solutions of the considered Pell equation.

METHOD OF ANALYSIS:

The Pell equation to be solved is \( x^2 = 15y^2 - 11^t, t = 2m + 1 \) (1)

First, we consider the Pell equation \( x^2 = 15y^2 - 11 \) (2)
whose fundamental solution is \((\tilde{x}_0, \tilde{y}_0) = (2,1)\).

The other solutions of (2) can be derived from the relations
\[
\tilde{x}_n = \frac{f_n}{2}, \quad \tilde{y}_n = \frac{g_n}{2\sqrt{15}}
\]
where
\[
f_n = [(2 + \sqrt{15})^{n+1} + (2 - \sqrt{15})^{n+1}]
\]
\[
g_n = [(2 + \sqrt{15})^{n+1} - (2 - \sqrt{15})^{n+1}]
\]
Now, we consider the general equation

\[ x^2 = 15y^2 - 11^{2m+1}, \quad m \geq 1 \]  

The initial solution of (3) is

\[ X_1 = 13 \cdot 11^{m-1}, \quad Y_1 = 10 \cdot 11^{m-1} \]

Applying the lemma of Brahmagupta between \((X_1, Y_1)\) and the solutions of the classical pell equation \(x^2 = 15y^2 + 1\), the other solutions of (3) can be obtained from the relations

\[
X_{n+2} = 11^{m-1}\left[\frac{13f_n}{2} + \frac{150g_n}{2\sqrt{15}}\right]
\]

\[
Y_{n+2} = 11^{m-1}\left[\frac{10f_n}{2} + \frac{13g_n}{2\sqrt{15}}\right]
\]

The recurrence relations satisfied by the solution of (1) are found to be

\[
X_{n+4} - 8X_{n+3} + X_{n+2} = 0
\]

\[
Y_{n+4} - 8Y_{n+3} + Y_{n+2} = 0
\]

\[
X_1 = \begin{cases} 2 & \text{if } m = 0 \\ 13 \cdot 11^{m-1} & \text{if } m \geq 1 \end{cases}
\]

\[
X_2 = \begin{cases} 19 & \text{if } m = 0 \\ 202 \cdot 11^{m-1} & \text{if } m \geq 1 \end{cases}
\]

and

\[
Y_1 = \begin{cases} 1 & \text{if } m = 0 \\ 10 \cdot 11^{m-1} & \text{if } m \geq 1 \end{cases}
\]

\[
Y_2 = \begin{cases} 4 & \text{if } m = 0 \\ 53 \cdot 11^{m-1} & \text{if } m \geq 1 \end{cases}
\]

A few interesting properties satisfied by the solutions of (1) are exhibited below:

(i) \(Y_{n+3} = X_{n+2} + 4Y_{n+2}\)

(ii) \(X_{n+3} = 4X_{n+2} + 15Y_{n+2}\)

(iii) \(Y_{n+4} = 8X_{n+2} + 31Y_{n+2}\)

(iv) \(X_{n+4} = 31X_{n+2} + 120Y_{n+2}\)

(v) \(Y_{n+4} = X_{n+3} + 4Y_{n+3}\)

(vi) \(X_{n+4} = 4X_{n+3} + 15Y_{n+3}\)

(vii) \(X_{n+4}^2 - X_{n+2}^2 = 240X_{n+3}Y_{n+3}\)

Each of the following triples forms an A.P

(a) \((X_{n+2}, 4X_{n+3}, X_{n+4})\)

(b) \((X_{n+3}, 2Y_{n+3}, Y_{n+2})\)

(c) \((X_{n+2}, 15Y_{n+3}, X_{n+4})\)
1. Define \( r = \frac{X_{n+2} + Y_{n+2}}{2} \) and \( s = \frac{Y_{n+2}}{2} \), where \((X_{n+2},Y_{n+2})\) is any solution of (1). Note that \( r, s \) are integers and \( r > s > 0 \). Treat \( r \) and \( s \) as the generators of the Pythagorean triangle \( T(\alpha,\beta,\gamma) \), where \( \alpha = 2rs, \beta = r^2 - s^2, \gamma = r^2 + s^2 \). Let \( A \) and \( P \) represented its area and perimeter respectively. Then, this Pythagorean triangle \( T \) is such that

\[
(a) 30\beta - \alpha - 29\gamma \equiv 0 \pmod{11^t}
\]

\[
(b) \gamma - 31\alpha + \frac{120A}{P} \equiv 0 \pmod{11^t}
\]

2. Let \( x \) and \( y \) be taken as the sides of a rectangle \( R \) whose length of the diagonal, perimeter and area are denoted by \( L \), \( P \) and \( A \) respectively. Note that

\[
(i) 6[L^2 + 11^t] \text{ is a nasty number}
\]

\[
(ii) P^2 - 8A = 4L^2
\]

**CONCLUSION**

In this paper, the integer solutions of the Pell equation \( x^2 = 15y^2 - 11^t \) where \( t \) odd are obtained. For the case \( t \) even, we find that there is no integer solution as the negative Pell equation \( x^2 = 15y^2 - 1 \) has no integer solution. To conclude, one may search for integer solutions of other choices of negative Pell equations

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