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Research Article

Integral Solution of the Non-Homogeneous Heptic Equation with Five Unknowns $x^4+y^4-(x-y)z^3=2(k^2+6s^2)w^2T^5$

S. Vidhyalakshmi[1], K. Lakshmi[2*] M.A. Gopalan[3]
[1,2,3]Department of Mathematics, Shrimati Indira Gandhi College,Trichy-620002, Tamilnadu, India

***Corresponding author**
K. Lakshmi
Email: lakshmi16654@gmail.com

Abstract: We obtain infinitely many non-zero integer quintuples (x,y,z,w,T) satisfying the non-homogeneous equation of degree seven with five unknowns given by $x^4+y^4-(x-y)z^3=2(k^2+6s^2)w^2T^5$.Various interesting properties between the solutions and special numbers are presented.
Keywords: Non-homogeneous equation, integral solutions, special numbers

INTRODUCTION

The theory of diophantine equations offers a rich variety of fascinating problems. In particular, homogeneous and non-homogeneous equations of higher degree have aroused the interest of numerous Mathematicians since antiquity [1-3].Particularly in [4, 5] special equations of sixth degree with four and five unknowns are studied. In [6-9] heptic equations with three and five unknowns are analysed.This paper concerns with the problem of determining non-trivial integral solution of the non- homogeneous equation of seventh degree with five unknowns given by, $x^4+y^4-(x-y)z^3=2(k^2+6s^2)w^2T^5$ .A few relations between the solutions and the special numbers are presented.

METHOD OF ANALYSIS

The Diophantine equation representing the non- homogeneous equation of degree seven is given by

$$x^4+y^4-(x-y)z^3=2(k^2+6s^2)w^2T^5 \tag{1}$$

Introduction of the transformations

$$x=w+z,\ y=w-z \tag{2}$$

in (1) leads to

$$w^2+6z^2=(k^2+6s^2)T^5 \tag{3}$$

The above equation (3) is solved through different approaches and thus, one obtains different sets of solutions to (1)

Case1: k^2+6s^2 is not a perfect square

Approach1:

Let $T=a^2+6b^2$ (4)

Substituting (4) in (3) and using the method of factorisation, define

$$(w+i\sqrt{6}z)=(k+i\sqrt{6}s)(a+i\sqrt{6}b)^5 \tag{5}$$

Equating real and imaginary parts in (5) we get

$$\left.\begin{aligned} w&=kf(a,b)-6sg(a,b)\\ z&=sf(a,b)+kg(a,b)\end{aligned}\right\} \tag{6}$$

where

$$\left.\begin{array}{l} f(a,b) = a^5 - 60a^3b^2 + 180ab^4 \\ g(a,b) = 5a^4b - 60a^2b^3 + 36b^5 \end{array}\right\} \quad (7)$$

In view of (2) and (4), the corresponding values of x, y, z, w and T are represented by

$$\left.\begin{array}{l} x = (k+s)f(a,b) + (k-6s)g(a,b) \\ y = (k-s)f(a,b) - (k+6s)g(a,b) \\ z = sf(a,b) + kg(a,b) \\ w = kf(a,b) - 6sg(a,b) \\ T = a^2 + 6b^2 \end{array}\right\} \quad (8)$$

Properties:

1. The following expressions are nasty numbers

$$(a)\ 6(k^2+6s^2)(kz(a,1) - sw(a,1)) - 6(k^2+6s^2)^2$$

$$(120F_{4,a,3} - 180P_a^3 - 50T_{3,a} + 195T_{4,a} - 130T_{5,a})$$

$$(b)\ (k-6s)\{y(a,a) - (k-s)[120F_{5,a,3} - 240F_{4,a,3} - 35CP_{a,6} + 120T_{3,a} +$$

$$156(3T_{4,a} - 2T_{5,a})] - (k-6s)[10T_{3,a}.T_{4,a} - 5CP_{a,6} - 120T_{3,a} + 60(3T_{4,a} - 2T_{5,a})]\}$$

2. $w(a,a) - (121k + 114s)[120F_{5,a,3} - 240F_{4,a,3} + 150P_a^3 - 30T_{3,a} + 2CP_{a,3} - CP_{a,6}] = 0$

3. $z(a,a) = (121s - 19k)[120F_{5,a,3} - 60F_{4,a,5} - 30P_a^3 + 30T_{3,a} + 3T_{4,a} - 2T_{5,a}]$

4. $x(a,a) + y(a,a) + T(a,a) - 2w(a,a) - (6p_a^3 - CP_{a,6} - 4T_{3,a}) \equiv 0 \pmod 6$

5. $x(a,a) - (k+s)[120F_{5,a,3} - 240F_{4,a,3} - 35CP_{a,6} + 120T_{3,a} + 156(3T_{4,a} - 2T_{5,a})] +$

$$(k-6s)[10T_{3,a}.T_{4,a} - 5CP_{a,6} - 120T_{3,a} + 60(3T_{4,a} - 2T_{5,a}) + 36] = 0$$

6. $T(2^{2n}, 2^{2n}) = 7(KY_{2n} - J_{2n+1})$

7. $T(a,a) + y(a,a) - x(a,a) - T_{16,a} + (242s - 38k)$

$$[4T_{3,a}.CP_{a,3} - T_{4,a^2} - 6P_a^3 + 4T_{3,a}] \equiv 0 \pmod 6.$$

8. $T(a(a+1), a+1) - (2T_{3,a^2} + 2CP_{a,6} + S_a + 54T_{4,a} - 36T_{5,a}] \equiv 0 \pmod 5$

9. $2z(a,a) + T(a,a) - (121s - 19k)[3T_{4,a}.CP_{a,4} - CP_{a,6}] - 2T_{9,a} \equiv 0 \pmod 5$

Approach2:

Now, rewrite (3) as, $w^2 + 6z^2 = (k^2 + 6s^2)T^5 * 1$ (9)

Also 1 can be written as

$$1 = \frac{(5+i2\sqrt{6})(5-i2\sqrt{6})}{7^2} \quad (10)$$

Substituting (4) and (10) in (9) and using the method of factorisation, define,

$$(w + i\sqrt{6}z) = \frac{(5+i2\sqrt{6}s)}{7}(k + i\sqrt{6}s)(a + i\sqrt{6}b)^5 \quad (11)$$

Following the same procedure as in approach1 we get the integral solution of (1) as

$$\left.\begin{array}{l} x=7^5[(k-s)f(A,B)-(k+6s)g(A,B)] \\ y=7^4(3k-17s)f(A,B)-(17k+18s)g(A,B) \\ z=7^4(2k+5s)f(A,B)+(5k-12s)g(A,B) \\ w=7^4(5k-12s)f(A,B)-6(2k+5s)g(A,B) \\ T=7^2(A^2+6B^2) \end{array}\right\} \quad (12)$$

Approach3:
1 can also be written as

$$1=\frac{(1+i2\sqrt{6})(1-i2\sqrt{6})}{5^2} \quad (13)$$

Following the same procedure as in approach1 we get the integral solution of (1) as

$$\left.\begin{array}{l} x=5^4[(3k-11s)f(A,B)-(11k+18s)g(A,B)] \\ y=-5^4(k+13s)f(A,B)+(13k-6s)g(A,B) \\ z=5^4(2k+s)f(A,B)+(k-12s)g(A,B) \\ w=5^4(k-12s)f(A,B)-6(2k+s)g(A,B) \\ T=5^2(A^2+6B^2) \end{array}\right\} \quad (14)$$

Approach4:
1 can also be written as

$$1=\frac{(6-\alpha^2+i2\alpha\sqrt{6})(6-\alpha^2-i2\alpha\sqrt{6})}{(6+\alpha^2)^2} \quad (15)$$

Following the same procedure as above we get the integral solution of (1) as

$$\left.\begin{array}{l} x=(6+\alpha^2)^4[\{(6-\alpha^2)(k+s)+2\alpha(k-6s)\}f(A,B)+\{(6-\alpha^2)(k-6s)-12\alpha(k+s)\}g(A,B)] \\ y=(6+\alpha^2)^4[\{(6-\alpha^2)(k-s)-2\alpha(k+6s)\}f(A,B)-\{(6-\alpha^2)(k+6s)+12\alpha(k-s)\}g(A,B)] \\ z=(6+\alpha^2)^4[\{(6-\alpha^2)s+2k\alpha\}f(A,B)+\{(6-\alpha^2)k-12\alpha s\}g(A,B)] \\ w=(6+\alpha^2)^4[\{(6-\alpha^2)k-12\alpha s\}f(A,B)-\{6(6-\alpha^2)s+2k\alpha\}g(A,B)] \\ T=(6+\alpha^2)^2(A^2+6B^2) \end{array}\right\}$$

(16)

Approach5:
Assuming

$$T=(k^2+6s^2)\overline{T},\ w=(k^2+6s^2)^3\overline{T}^2W,\ z=(k^2+6s^2)^3\overline{T}^2Z \quad (17)$$

in (3), it reduces to

$$W^2+6Z^2=\overline{T} \quad (18)$$

Taking $\overline{T}=t^2$ in (18) & solving and using (17) we get

$$\left.\begin{array}{l} w=(k^2+6s^2)^3(\alpha^2+6\beta^2)^4(\alpha^2-6\beta^2) \\ z=(k^2+6s^2)^3 2\alpha\beta(\alpha^2+6\beta^2)^4 \\ T=(k^2+6s^2)(\alpha^2+6\beta^2)^2 \end{array}\right\} \quad (19)$$

Using (19) and (2), the corresponding integral solutions to (1) can be obtained.

Approach6:
Assuming

$$w = WT^2, \quad z = ZT^2 \tag{20}$$

in (3), we get, $W^2 + 6Z^2 = (k^2 + 6s^2)T$ (21)

Taking $T = t^2$ in (21) and arranging we have

$$(W - kt)(W + kt) = 6(st - Z)(st + Z) \tag{22}$$

Writing (21) as a system of double equations and solving, we get

$$W = (12s - k)t_1, \quad Z = (s + 2k)t_1, \quad t = 5t_1 \tag{23}$$

Using (20), (23) and (2) the corresponding integral solution can be obtained.

Case2: $k^2 + 6s^2$ **is a perfect square**

Choose k and s such that $k^2 + 6s^2 = d^2$. (24)

Substituting (24) in (3) we get

$$w^2 + 6z^2 = d^2 T^5 \tag{25}$$

Approach7:

Assuming $w = dWT^2, \quad z = dZT^2$ (26)

in (25), we get $W^2 + 6Z^2 = T$ (27)

$$T = (a^2 + 6b^2)^n \tag{28}$$

Substituting (28) in (27) and writing it as a system of double equations and solving we get

$$\left.\begin{aligned} W &= \frac{1}{2}[(a + i\sqrt{6}b)^n + (a - i\sqrt{6}b)^n] \\ Z &= \frac{1}{2i\sqrt{6}}[(a + i\sqrt{6}b)^n - (a - i\sqrt{6}b)^n] \end{aligned}\right\} \tag{29}$$

Using (29), (28), (26) and (2) we get the integral solution to (1) as

$$\left.\begin{aligned} x &= \frac{d}{2}(a + 6b^2)^{2n}[f(a,b) - \frac{i}{\sqrt{6}}g(a,b)] \\ y &= \frac{d}{2}(a + 6b^2)^{2n}[f(a,b) + \frac{i}{\sqrt{6}}g(a,b)] \\ Z &= \frac{d}{2i\sqrt{6}}(a + 6b^2)^{2n} g(a,b) \\ W &= \frac{d}{2}(a + 6b^2)^{2n} f(a,b) \\ T &= (a + 6b^2)^n \end{aligned}\right\} \tag{30}$$

where

$$f(a,b) = (a + i\sqrt{6}b)^n + (a - i\sqrt{6}b)^n$$

$$g(a,b) = (a + i\sqrt{6}b)^n - (a - i\sqrt{6}b)^n$$

Properties:

1. $2^{n+1}.w(a,a) = GL_n(4,-28).d.7^{2n}(2P^5_{a^n}.T_{4,a^n} - CP_{a^n,6})$

2. $2^{n+1}.z(a,a)=GF_n(4,-28).d.7^{2n}(24F_{4,a^3.3}-6CP_{a^n,6}-22T_{3,a^n}$
$+5(3T_{4,a^n}-2T_{5,a^n})(2CP_{a^n,3}-CP_{a^n,6})$

3. $T(a,a)-7(2P^5_{a^n}-CP_{a^n,6})=0$

4. $4w(a,a)-7^{2n}.d.[(1+i\sqrt{6})^n+(1-i\sqrt{6})^n](SO_{a^n}.T_{4,a^n}+CP_{a^n,3})=0$

5. $7^n(S_{a^n}+6PR_{a^n}-6T_{4,a^n})-6T(a,a)\equiv 0(\bmod 7)$

Approach8:

Assuming $w=dWT,\quad z=dZT$ (31)

in (25), we get $W^2+6Z^2=T^3$ (32)

Then the solution to (32) is obtained as

$$\left.\begin{aligned} W&=p(p^2+6q^2)\\ Z&=q(p^2+6q^2)\\ T&=p^2+6q^2 \end{aligned}\right\} \quad (33)$$

Using (33), (31) and (2), we get the integral solution to (1) as

$$\left.\begin{aligned} x&=d(p^2+6q^2)^2(p+q)\\ y&=d(p^2+6q^2)^2(p-q)\\ z&=qd(p^2+6q^2)^2\\ w&=pd(p^2+6q^2)^2\\ T&=p^2+6q^2 \end{aligned}\right\} \quad (34)$$

Properties:

1. $x(2a,a)+y(2a,a)=400d[120F_{5,a,3-}240F_{4,a,3}+50P^5_a+70T_{3,a}+2CP_{a,6}-SO_a]$

2. $x(a,a)-y(a,a)=200d[120F_{5,a,3-}60F_{4,a,5}-30P^3_a-15T_{4,a}-42(OH_a)+28CP_{a,6}]$

3. $2w(a,a)+z(a,a)=49d[4P^5_a.CP_{3,a}-6T_{4,a^2}-3CP_{a,6}-4P^5_a]$

4. $294J_{2n}-42T(2^{2n},2^{2n})$ is a nasty number

5. $2w(a,1)-d[6(OH)_a.T_{3,a}-12F_{4,a,5}+29CP_{a,6}+3T_{4,a}]\equiv 0(\bmod 72)$

Remark:

The solution to (32) can also be obtained as

$$\left.\begin{aligned} w&=d(\alpha^3-18\alpha\beta^2)(\alpha^2+6\beta^2)\\ z&=d(3\alpha^2\beta-6\beta^3)(\alpha^2+6\beta^2)\\ T&=\alpha^2+6\beta^2 \end{aligned}\right\} \quad (35)$$

Using (35), (31) and (2), the integral solution to (1) can be obtained.

Approach9:

Taking $T=t^2$ (36)

in (27) and solving, the integral solution to (1) can be obtained as

$$\left.\begin{aligned} x &= d(\alpha^2 - 6\beta^2 + 2\alpha\beta)(\alpha^2 + 6\beta^2)^4 \\ y &= d(\alpha^2 - 6\beta^2 - 2\alpha\beta)(\alpha^2 + 6\beta^2)^4 \\ w &= d(\alpha^2 - 6\beta^2)(\alpha^2 + 6\beta^2)^4 \\ z &= 2\alpha\beta . d(\alpha^2 + 6\beta^2)^4 \\ T &= (\alpha^2 + 6\beta^2)^2 \end{aligned}\right\} \quad (37)$$

The solutions in all the above approaches satisfy the following properties:

1. $xy - w^2 + z^2 = 0$
2. $xy - y^2 + 4wz = 0$
3. The following expressions are nasty numbers:
   (a). $x^2 + y^2 - 4xy + 2w^2$
   (b). $(x + y)^2 + 2w^2$
   (c). $2w(x + y + w)$
4 The following expressions are cubical integers:
   (a). $4(x^3 + y^3 - 6wz^2)$
   (b). $4(x^3 - y^3 - 6zw^2)$
   (c). $\frac{x^4 - y^4}{w} - 8zw^2$
5. $8(x^4 + y^4 - 2z^4 - 12z^2w^2)$ is a biquadratic integer
6. $z(x^3 + y^3)(3w^2 + z^2) - w(x^3 - y^3)(w^2 + 3z^2) = 0$

CONCLUSION

In conclusion, one may search for different patterns of solutions to (1) and their **c**orresponding properties.

MSC 2000 Mathematics subject classification: 11D41.

Notations:

$T_{m,n}$ -Polygonal number of rank n with size m

P_n^m - Pyramidal number of rank n with size m

SO_n -Stella octangular number of rank n

S_n -Star number of rank n

PR_n - Pronic number of rank n

OH_n - Octahedral number of rank n

J_n -Jacobsthal number of rank of n

j_n - Jacobsthal-Lucas number of rank n

KY_n -keynea number of rank n

$CP_{n,3}$ - Centered Triangular pyramidal number of rank n

$CP_{n,6}$ - Centered hexagonal pyramidal number of rank n

$F_{5,n,3}$ -Five Dimensional Figurative number of rank n whose generating polygon is a triangle.

$F_{4,n,3}$ -Four Dimensional Figurative number of rank n whose generating polygon is a triangle

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