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Research Article

On the cubic Equation with four unknowns $x^3 + y^3 = (z+w)^2(z-w)$

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Abstract: The sequences of integral solutions to the cubic equation with four variables $x^3 + y^3 = (z+w)^2(z-w)$ are obtained. A few properties among the solutions are also presented.
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INTRODUCTION

The Diophantine equations offer an unlimited field for research due to their variety [1-2] In particular, one may refer [3-14]for cubic equation with three unknowns. In [15-17] cubic equations with four unknowns are studied for its non-trival integral solution.This communication concerns with the problem of obtaining non-zero integral solutions of the cubic equation with four variables is given by $x^3 + y^3 = (z+w)^2(z-w)$. A few properties among the solutions and special numbers are presented.

Notations:

$$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$$

$$P_n^m = \frac{n(n+1)}{6}\left[(m-2)n + (5-m)\right]$$

$$PR_n = n(n+1)$$

$$Gno_n = 2n - 1$$

$$S_n = 6n(n-1) + 1$$

$$SO_n = n(2n^2 - 1)$$

$$j_n = 2^n + (-1)^n$$

$$J_n = \frac{1}{3}\left[2^n + (-1)^n\right]$$

$$CP_n^6 = -n^3$$

$$CP_n^8 = \frac{n(4n^2 - 1)}{3}$$

$$CP_n^9 = \frac{n(3n^2 - 1)}{2}$$

$$CP_n^{14} = \frac{n(7n^2 - 4)}{3}$$

$$CP_n^{16} = \frac{n(8n^2 - 5)}{3}$$

$$F_{4,n,4} = \frac{n(n+1)^2(n+2)}{6}$$

$$F_{4,n,6} = \frac{n^2(n+1)(n+2)}{6}$$

METHOD OF ANALYSIS

The cubic diophantine equation with four unknowns to be solved for getting non-zero integral solutions is

$$x^3 + y^3 = (z+w)^2(z-w) \qquad (1)$$

It is noted that is to noted that $(2k+4, -2k+2, k^2+k+4, -k^2-k+2)$ where k is an integer is a solution of the given problem. However, we have other patterns of solutions to (1) which are discussed below.

On substituting the linear transformations

x=u+v, y=u-v, z=u+p, w=u-p, $u \neq p$, $v \neq p$, (2)

in (1), it leads to

$$=(u-2p)^2 + 3v^2 = 4p^2 \qquad (3)$$

Five different patterns of integral solutions to (1) through solving (3) are illustrated as follows:

Pattern 1:

Equation (3) is satisfied by

$$\left.\begin{aligned} u - 2p &= 3a^2 - b^2 \\ v &= 2ab \\ p &= \frac{3a^2 + b^2}{2} \end{aligned}\right\} \qquad (4)$$

Since our aim is to find integral solutions both ‘a’ and ‘b’ should be either even (or) odd.

Case i: Suppose both a and b are even

Let a=2A and b=2B

Substituting the values of a and b in (4) and simplifying, we get

$$u = 24A^2,$$
$$v = 8AB,$$
$$p = 6A^2 + 2B^2$$

In view (2), the non-zero distinct integral solutions to (1) are given by

$$x = 8(3A^2 + AB)$$
$$y = 8(3A^2 - AB)$$
$$z = 2(15A^2 + B^2)$$
$$w = 2(9A^2 - B^2)$$

Properties:

1) $4[z(A,1) - y(A,1) - 4t_{5,A} + 6CP_A^{16} - 2]$ is a cubical integer.
2) Each of the following is a nasty number
   (i) $2(x(A,B) + y(A,B))$
   (ii) $2(z(A,B) + w(A,B))$
3) $x(A,9A) + z(A,9A) + w(A,9A)$ is a perfect square
4) $x(A,1) - y(A,1) + 8$ is written as 8 times difference of consecutive squares
5) $36[x(A,1)y(A,1) + 64t_{4,A}]$ is a biquadratic integer

Case ii: Suppose both 'a' and 'b' are odd.
Let a=2A+1, b=2B+1. Proceeding as in case (i) the non-zero distinct integral solutions to (1) are

$$x = 4(6A^2 + 2AB + 7A + B + 2)$$
$$y = 4(6A^2 - 2AB + 5A - B + 1)$$
$$z = 2(15A^2 + B^2 + 15A + B + 4)$$
$$w = 2(9A^2 - B^2 + 9A - B + 2)$$

Properties:

1) $2x(2A,A) - z(2A,A) - w(2A,A) - 8t_{10,A} \equiv 4(\mathrm{mod}\,48)$
2) $\dfrac{x(A+1,A)}{4} + 14(t_{6,A} - 2t_{4,A}) - 13$ is 2 times an odd square
3) $2(x(A,B) + y(A,B))$ and $2(z(A,B) + w(A,B))$ is a nasty number
4) $3z(A,1) - 5w(A,1) - 36 = 0$
5) $z(A,B) - w(A,B) + 24P_{B-1}^3 = 2[6PR_A + 4P_B^5]$

Pattern 2:
In (3) take

$$p = a^2 + 3b^2 \qquad (5a)$$

and write '4' as

$$4=(1+i\sqrt{3})(1-i\sqrt{3}) \qquad (5b)$$

Substituting (5a) and (5b) in (3) and employing the method of factorization, define

$$u - 2p + i\sqrt{3}v = (1 + i\sqrt{3})(a + i\sqrt{3}b)^2$$

Equating real and imaginary parts on both sides we get

$$u - 2p = a^2 - 6ab - 3b^2 \qquad (6)$$
$$v = a^2 + 2ab - 3b^2 \qquad (7)$$

Substituting (5a) in (6) we get

$$u = 3a^2 + 3b^2 - 6ab \qquad (8)$$

From (5a),(7),(8) and (2) the distinct integral solutions to (1) are expressed by

$$x(a,b) = 4(a^2 - ab)$$
$$y(a,b) = 2(a^2 + 3b^2 - 4ab)$$
$$z(a,b) = 2(2a^2 - 3ab + 3b^2)$$
$$w(a,b) = 2(a^2 - 3ab)$$

Properties:

1) $z(a,1) + y(a,1) - 6 = 2(5t_{5,a} - 3Gno_a)$
2) (i) $2[3x(a,b) - 2w(a,b)]$ is a perfect square
   (ii) $2(z(a,b) - w(a,b) + 2x(a,b) - y(a,b))$ is a perfect square
3) $x(1,b) - y(1,b) - z(1,b) + w(1,b) + 2S_b \equiv 2(\mathrm{mod}\, 8b)$
4) (i) $3(x(a,1)w(a,1) + 8a^2)$ is a nasty number
   (ii) $2(x + y + z + w)$ is a nasty number
5) $x(a,1)w(a,1) = 8[6F_{4,a,6} - 3CP_a^{14} + 2t_{3,a} + 5t_{6,a} - 10t_{4,a}]$

Pattern 3:
Instead of (5b), write '4' as

$$4 = \frac{(2 + 8i\sqrt{3})(2 - 8i\sqrt{3})}{7^2}$$

Proceeding as in Pattern 2 and replacing a by 7A and b by 7B, the corresponding non-zero distinct integer solutions to (1) are given by

$$x(A,B) = 14(12A^2 - 22AB + 6B^2)$$
$$y(A,B) = 14(4A^2 - 26AB + 30B^2)$$
$$z(A,B) = 7(23A^2 - 48AB + 57B^2)$$
$$w(A,B) = 7(9A^2 - 48AB + 15B^2)$$

Properties

1) $x(A, A(A+1)) = 28(24F_{4,A,5} - 6CP_A^5)$
2) $\frac{y(A, A+1) - x(A, A+1)}{112} - t_{5,A} \equiv 3(\mathrm{mod}\, 6A)$
3) $w(2^{2n},1) = 21(3j_{4n} - 48J_{2n} - 14)$
4) (i) $3z(A,1) - 3w(A,1) - 882$ is a nasty number
   (ii) $84(x(A,B) + y(A,B))$ is a nasty number

Pattern 4:
Taking

u-2p=2X, v=2V (9)

in (3), it becomes

$$X^2 + 3V^2 = p^2 \quad (9a)$$

which is satisfied by

$$X = a^2 - 3b^2, V = 2ab, p = a^2 + 3b^2 \quad (10)$$

In the view of (10),(9) and (2) the distinct integral solutions of (1) are given by

$x(a,b)=4(a^2+ab)$

$y(a,b)=4(a^2-ab)$

$z(a,b)=5a^2+3b^2$

$w(a,b)=3(a^2-b^2)$

Properties

1) $3(x(a,b)+y(a,b))$ and $3(z(a,b)+w(a,b))$ are nasty numbers.
2) (i) $z(a,a^2)+w(a,a^2)-2y(a,a^2)$ is a cubical integer.
   (ii) $x(a,a^2)+y(a,a^2)$ is a cubical integer.
3) $x(a,1)y(a,1)=16(12F_{4,a,4}-3CP_a^8-3aGno_{a+1})$
4) $x(a,1)y(a,1)-z(a,1)w(a,1)-9=6F_{4,a,6}-2CP_a^9-11t_{4,a}-PR_a$

Pattern 5:

Re-write (9a) as

$$p^2-3V^2=X^2*1 \qquad (11)$$

and write

$$\text{'1'}=(2+\sqrt{3})(2-\sqrt{3}),\ X=a^2-3b^2 \qquad (12)$$

Substituting (12) in (11) and using method of factorization, define

$(p+\sqrt{3}v)=(2+\sqrt{3})(a+\sqrt{3}b)^2$

Equating rational and irrational parts on both sides we obtain

$p=2(a^2+3b^2)+6ab$

$v=a^2+3b^2+4ab \qquad (13)$

$X=a^2=3b^2 \qquad (14)$

From (9),(13) and (14) we have

$u=6a^2+12ab+6b^2$

$v=2a^2+8ab+6b^2$

$p=2a^2+6ab+6b^2$

In view of (2) the non-zero distinct integral solutions to (1) are

$x=2(4a^2+10ab+6b^2)$

$y=2(2a^2+2ab)$

$z=2(4a^2+9ab+6b^2)$

$w=2(2a^2+3ab)$

Properties:

1) $x(a,a^2)+y(a,a^2)-z(a,a^2)-4CP_a^9-4PR_a\equiv 0(\bmod 2a)$,
2) (i) 6[3y(a,b) -2w(a,b)] is a nasty number.
   (ii) $2[z(1,b)-3w(1,b)+4]$ is a nasty number.
3) $z(a,1)(w(a,1)-y(a,1))=2[3CP_a^8+2t_{10,a}+26t_{3,a}-12t_{4,a}]$
4) $x(a(a+1),a)=2[48F_{4,a,4}+CP_a^4+9SO_a+18CP_a^6]$

CONCLUSION

One may search for other patterns of solutions and their corresponding properties.

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