Research Article

On the cubic Equation with four unknowns \( x^3 + y^3 = (z + w)^2 (z - w) \)

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Abstract: The sequences of integral solutions to the cubic equation with four variables \( x^3 + y^3 = (z + w)^2 (z - w) \) are obtained. A few properties among the solutions are also presented.

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INTRODUCTION
The Diophantine equations offer an unlimited field for research due to their variety [1-2] In particular, one may refer [3-14] for cubic equation with three unknowns. In [15-17] cubic equations with four unknowns are studied for its non-trivial integral solution. This communication concerns with the problem of obtaining non-zero integral solutions of the cubic equation with four variables is given by \( x^3 + y^3 = (z + w)^2 (z - w) \). A few properties among the solutions and special numbers are presented.

Notations:

\[ t_{m,n} = n \left\lceil \frac{1 + (n-1)(m-2)}{2} \right\rceil \]
\[ P_n^m = \frac{n(n+1)}{6} \left[(m-2)n + (5-m) \right] \]
\[ PR_n = n(n+1) \]
\[ Gno_n = 2n-1 \]
\[ S_n = 6n(n-1) + 1 \]
\[ SO_n = n(2n^2 - 1) \]
\[ j_n = 2^n + (-1)^n \]
\[ J_n = \frac{1}{3} \left[ 2^n + (-1)^n \right] \]
\[ CP_n^6 = -n^3 \]
\[ CP_n^8 = \frac{n(4n^2 - 1)}{3} \]
\[ CP_n^9 = \frac{n(3n^2 - 1)}{2} \]
\[ CP_n^{14} = \frac{n(7n^2 - 4)}{3} \]
\[ CP_n^{16} = \frac{n(8n^2 - 5)}{3} \]
\[ F_{4,n,4} = \frac{n(n + 1)^2(n + 2)}{6} \]
\[ F_{4,n,6} = \frac{n^2(n + 1)(n + 2)}{6} \]

**METHOD OF ANALYSIS**

The cubic diophantine equation with four unknowns to be solved for getting non-zero integral solutions is

\[ x^3 + y^3 = (z + w)^2(z - w) \]  \hspace{1cm} (1)

It is noted that is to noted that \((2k + 4, -2k + 2, k^2 + k + 4, -k^2 - k + 2)\) where \(k\) is an integer is a solution of the given problem. However, we have other patterns of solutions to (1) which are discussed below.

On substituting the linear transformations

\[ x = u + v, \quad y = u - v, \quad z = u + p, \quad w = u - p, \quad u \neq p, \quad v \neq p, \]  \hspace{1cm} (2)

in (1), it leads to

\[ \begin{align*}
(u - 2p)^2 + 3v^2 &= 4p^2 \\
\end{align*} \]  \hspace{1cm} (3)

Five different patterns of integral solutions to (1) through solving (3) are illustrated as follows:

**Pattern 1:**

Equation (3) is satisfied by

\[ \begin{align*}
\begin{cases}
u - 2p = 3a^2 - b^2 \\
v = 2ab \\
p = \frac{3a^2 + b^2}{2}
\end{cases}
\end{align*} \]  \hspace{1cm} (4)

Since our aim is to find integral solutions both ‘a’ and ‘b’ should be either even (or) odd.

**Case i:** Suppose both \(a\) and \(b\) are even

Let \(a = 2A\) and \(b = 2B\)

Substituting the values of \(a\) and \(b\) in (4) and simplifying, we get

\[ \begin{align*}
u &= 24A^2 \\
v &= 8AB \\
p &= 6A^2 + 2B^2
\end{align*} \]

In view (2), the non-zero distinct integral solutions to (1) are given by
\[ x = 8(3A^2 + AB) \]
\[ y = 8(3A^2 - AB) \]
\[ z = 2(15A^2 + B^2) \]
\[ w = 2(9A^2 - B^2) \]

**Properties:**

1) \( 4[z(A,1) - y(A,1) - 4t_{5,A} + 6CP_{A}^{16} - 2] \) is a cubical integer.

2) Each of the following is a nasty number
   (i) \( 2(x(A,B) + y(A,B)) \)
   (ii) \( 2(z(A,B) + w(A,B)) \)

3) \( x(A,9A) + z(A,9A) + w(A,9A) \) is a perfect square

4) \( x(A,1) - y(A,1) + 8 \) is written as 8 times difference of consecutive squares

5) \( 36[x(A,1)y(A,1) + 64t_{4,A}] \) is a biquadratic integer

**Case ii:** Suppose both ‘a’ and ‘b’ are odd.

Let \( a = 2A + 1 \), \( b = 2B + 1 \). Proceeding as in case (i) the non-zero distinct integral solutions to (1) are

\[ x = 4(6A^2 + 2AB + 7A + B + 2) \]
\[ y = 4(6A^2 - 2AB + 5A - B + 1) \]
\[ z = 2(15A^2 + B^2 + 15A + B + 4) \]
\[ w = 2(9A^2 - B^2 + 9A - B + 2) \]

**Properties:**

1) \( 2x(2A,A) - z(2A,A) - w(2A,A) - 8t_{10,A} \equiv 4(\text{mod } 48) \)

2) \( \frac{x(A+1,A)}{4} + 14(t_{6,A} - 2t_{4,A}) - 13 \) is 2 times an odd square

3) \( 2(x(A,B) + y(A,B)) \) and \( 2(z(A,B) + w(A,B)) \) is a nasty number

4) \( 3z(A,1) - 5w(A,1) - 36 = 0 \)

5) \( z(A,B) - w(A,B) + 24P_{b-1}^{3} = 2[6PR_{A} + 4P_{b}^{5}] \)

**Pattern 2:**

In (3) take

\[ p = a^2 + 3b^2 \]

and write ‘ 4’ as

\[ 4 = (1+i\sqrt{3})(1-i\sqrt{3}) \]

Substituting (5a) and (5b) in (3) and employing the method of factorization, define

\[ u - 2p + i\sqrt{3}v = (1 + i\sqrt{3})(a + i\sqrt{3}b)^2 \]

Equating real and imaginary parts on both sides we get

\[ u - 2p = a^2 - 6ab - 3b^2 \]  
(6)
\[ v = a^2 + 2ab - 3b^2 \]  
(7)

Substituting (5a) in (6) we get

\[ u = 3a^2 + 3b^2 - 6ab \]  
(8)

From (5a),(7),(8) and (2) the distinct integral solutions to (1) are expressed by
\[
x(a, b) = 4(a^2 - ab) \\
y(a, b) = 2(a^2 + 3b^2 - 4ab) \\
z(a, b) = 2(2a^2 - 3ab + 3b^2) \\
w(a, b) = 2(a^2 - 3ab)
\]

Properties:

1) \( z(a,1) + y(a,1) - 6 = 2(5t_{s,a} - 3Gno_{a,a}) \)

2) (i) \( 2[3x(a, b) - 2w(a, b)] \) is a perfect square

(ii) \( 2(z(a, b) - w(a, b) + 2x(a, b) - y(a, b)) \) is a perfect square

3) \( x(1, b) - y(1, b) - z(1, b) + w(1, b) + 2S_{b} \equiv 2(\text{mod } 8b) \)

4) (i) \( 3(x(a,1)w(a,1) + 8a^2) \) is a nasty number

(ii) \( 2(x + y + z + w) \) is a nasty number

5) \( x(a,1)w(a,1) = 8[6F_{4,a,b} - 3CP_{a}^{14} + 2t_{3,a} + 5t_{6,a} - 10t_{4,a}] \)

Pattern 3:
Instead of (5b), write ‘4’ as

\[
4 = \frac{(2 + 8i\sqrt{3})(2 - 8i\sqrt{3})}{7^2}
\]

Proceeding as in Pattern 2 and replacing \( a \) by 7A and \( b \) by 7B, the corresponding non-zero distinct integer solutions to (1) are given by

\[
x(A, B) = 14(12A^2 - 22AB + 6B^2) \\
y(A, B) = 14(4A^2 - 26AB + 30B^2) \\
z(A, B) = 7(23A^2 - 48AB + 57B^2) \\
w(A, B) = 7(9A^2 - 48AB + 15B^2)
\]

Properties

1) \( x(A, A(A + 1)) = 28(24F_{4,A,5} - 6CP_{A}^{5}) \)

2) \( \frac{y(A, A + 1) - x(A, A + 1)}{112} - t_{5,a} \equiv 3(\text{mod } 6A) \)

3) \( w(2^{2n},1) = 21(3j_{2n} - 48J_{2n} - 14) \)

4) (i) \( 3z(A,1) - 3w(A,1) - 882 \) is a nasty number

(ii) \( 84(x(A, B) + y(A, B)) \) is a nasty number

Pattern 4:
Taking \( u - 2p = 2X, v = 2V \) (9)
in (3), it becomes

\[
X^2 + 3V^2 = p^2
\]

which is satisfied by

\[
X = a^2 - 3b^2, V = 2ab, p = a^2 + 3b^2
\]

In the view of (10), (9) and (2) the distinct integral solutions of (1) are given by
\[x(a, b) = 4(a^2 + ab)\]
\[y(a, b) = 4(a^2 - ab)\]
\[z(a, b) = 5a^2 + 3b^2\]
\[w(a, b) = 3(a^2 - b^2)\]

**Properties**

1) \(3(x(a, b) + y(a, b))\) and \(3(z(a, b) + w(a, b))\) are nasty numbers.

2) \[(i)\] \(z(a, a^2) + w(a, a^2) - 2y(a, a^2)\) is a cubical integer.
   \[\text{(ii)}\] \(x(a, a^2) + y(a, a^2)\) is a cubical integer.

3) \(x(a, l)y(a, l) = 16(12F_{4, a, 4} - 3CP^8_a - 3aGno_{a+1})\)

4) \(x(a, l)y(a, l) - z(a, l)w(a, l) - 9 = 6F_{4, a, 6} - 2CP^9_a - 11t_{4, a} - PR_a\)

**Pattern 5:**

Re-write (9a) as

\[p^2 - 3v^2 = X^2 \equiv 1 \pmod{1}\]  \hfill (11)

and write

\[\text{‘1’ = } (2 + \sqrt{3})(2 - \sqrt{3}), \quad X = a^2 - 3b^2\]  \hfill (12)

Substituting (12) in (11) and using method of factorization, define

\[(p + \sqrt{3}v) = (2 + \sqrt{3})(a + \sqrt{3}b)^2\]

Equating rational and irrational parts on both sides we obtain

\[p = 2(a^2 + 3b^2) + 6ab\]
\[v = a^2 + 3b^2 + 4ab\]
\[X = a^2 = 3b^2\]  \hfill (13)

From (9), (13) and (14) we have

\[u = 6a^2 + 12ab + 6b^2\]
\[v = 2a^2 + 8ab + 6b^2\]
\[p = 2a^2 + 6ab + 6b^2\]

In view of (2) the non-zero distinct integral solutions to (1) are

\[x = 2(4a^2 + 10ab + 6b^2)\]
\[y = 2(2a^2 + 2ab)\]
\[z = 2(4a^2 + 9ab + 6b^2)\]
\[w = 2(2a^2 + 3ab)\]

**Properties:**

1) \(x(a, a^2) + y(a, a^2) - z(a, a^2) - 4CP^9_a - 4PR_a \equiv 0 \pmod{2a}\).

2) \[(i)\] \(6[3y(a, b) - 2w(a, b)]\) is a nasty number.
   \[\text{(ii)}\] \(2[z(1, b) - 3w(1, b) + 4]\) is a nasty number.

3) \(z(a, l)(w(a, l) - y(a, l)) = 2[3CP^8_a + 2t_{10, a} + 26t_{3, a} - 12t_{4, a}]\)

4) \(x(a + 1, a) = 2[48F_{4, a, 4} + CP^4_a + 9SO^9_a + 18CP^6_a]\)
CONCLUSION

One may search for other patterns of solutions and their corresponding properties.

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