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Research Article

A note on p-valently close-to-convex functions

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Abstract: A theorem involving multivalent close-to-convex functions is considered and then its certain consequences are given.
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INTRODUCTION

Let $A_n(p)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{p+k}z^{p+k}(n,p\in N=\{1,2,\cdots\}), \qquad (1)$$

which are analytic and p-valent in the open unit disk $U=\{z:|z|<1\}$.

A function $f\in A_n(p)$ is said to be p-valently starlike functions of order α $(0\le\alpha<p)$in U if it satisfies the following inequality:

$$Re\{\frac{zf'(z)}{f(z)}\}>\alpha. \qquad (2)$$

We denote this class by $S_n^*(p,\alpha)$.

Similarly, a function $f\in A_n(p)$ is said to be p-valently convex functions of order α $(0\le\alpha<p)$in U if

$$Re\{1+\frac{zf''(z)}{f'(z)}\}>\alpha. \qquad (3)$$

It follows from expression (2) and (3) that f is convex if and only if, zf' is starlike.A function $f\in A_n(p)$ is said to be close-to-convex functions of order α $(0\le\alpha<p)$ in U if

$$Re\{\frac{f'(z)}{z^{p-1}}\}>\alpha. \qquad (4)$$

We denote by $C_n(p,\alpha)$.

A function $f\in A_n(p)$ is said to belong to the class of p-valently close-to-convex functions of order α and type ξ in U, if there exists a function $g(z)\in S_n^*(p,\xi)$ such that

$$Re\{\frac{zf'(z)}{g(z)}\}>\alpha, 0\le\alpha,\xi<p, z\in U. \qquad (5)$$

We denote the class of all such functions by $K_p(\alpha,\xi)$.

The class $S_n^*(p,0)$ was introduced by Goodman[1], whereas Patil and Thakare[2] generalized this idea to get the class $S_n^*(p,\alpha)$.Owa[3] introduced the class $C_n(p,\alpha)$,also $C_n(p,0)$ was introduced by Goodman[2].The class $K_p(\alpha,\xi)$ was studied by Aouf[4] and the class $K_1(\alpha,\xi)$ was studied by Libera[5].
The following lemma (popularly known as Jack's lemma) will be required in our present investigation.

Lemma 1. (see [6,7]) Let the (nonconstant) function $w(z)$ be analytic in U with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point $z_0\in U$, then

$$z_0 w'(z_0)=kw(z_0), \tag{6}$$

where k is a real number and $k\geq 1$.

MAIN RESULTS AND THEIR CONSEQUENCES

Theorem 1. Let $f\in A_n(p), w\in C$, $\{0\}, 0\leq\alpha,\xi<p, p\in N, z\in U,$ and also let the function H be defined by

$$H(z)=\frac{zf'(z)}{zf'(z)-pg(z)}[1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}], \tag{7}$$

where $g\in S_n^*(p,\xi)$. If $H(z)$ satisfies one of the following conditions:

$$Re\{H(z)\}=\Big\{<|w|^{-2}\ Re\{w\} if\ Re\{w\}>0,(8)\neq 0 if\ Re\{w\}=0,(9)>|w|^{-2}\ Re\{w\} if\ Re\{w\}<0.(10)$$

or

$$Im\{H(z)\}=\Big\{<|w|^{-2}\ Im\{\overline{w}\} if\ Im\{\overline{w}\}>0,(11)\neq 0 if\ Im\{\overline{w}\}=0,(12)>|w|^{-2}\ Im\{\overline{w}\} if\ Im\{\overline{w}\}<0.(13)$$

then

$$|(\frac{zf'(z)}{g(z)}-p)^w|<p-\alpha, \tag{14}$$

where the value of complex power in (10) is taken to be as its principal value.
Proof.. We define the function Ω by

$$(\frac{zf'(z)}{g(z)}-p)^w=(p-\alpha)\Omega(z), \tag{15}$$

where $w\in C$, $\{0\}, 0\leq\alpha,\xi<p, p\in N, z\in U, f\in H_n(p)$ and $g\in S_n^*(p,\xi)$.
We see clearly that the function Ω is regular in U and $\Omega(0)=0$. Making use of the logarithmic differentiation of both sides of (11) with respect to the known complex variable z, we can get

$$wz(\frac{zf'(z)}{g(z)}-p)^{-1}(\frac{zf'(z)}{g(z)}-p)'=\frac{z\Omega'(z)}{\Omega(z)}, \tag{16}$$

and if we make use of equality (11) once again, we can find that

$$H(z)=\frac{\overline{w}}{|w|^2}\frac{z\Omega'(z)}{\Omega(z)}, w\in C,\ \{0\}, z\in U. \tag{17}$$

Assume that there exists a point $z_0\in U$ such that

$$\max_{|z|<|z_0|}|\Omega(z)|=|\Omega(z_0)|=1, z\in U. \tag{18}$$

Applying Lemma 1, we can obtain

$$z_0\Omega'(z_0)=c\Omega(z_0), c\geq 1. \tag{19}$$

Then (15) yields

$$Re\{H(z_0)\}=Re\{\frac{\overline{w}}{|w|^2}\frac{z_0\Omega'(z_0)}{\Omega(z_0)}\}=Re\{c\overline{w}|w|^{-2}\}, \tag{20}$$

so that

$$Re\{H(z_0)\}=\Big\{\geq|w|^{-2}\ Re\{w\} if\ Re\{w\}>0,(21)=0 if\ Re\{w\}=0,(22)\leq|w|^{-2}\ Re\{w\} if\ Re\{w\}<0,(23)$$

or

$$Im\{H(z_0)\} = \{\geq |w|^{-2}\ Im\{\overline{w}\} if\ Im\{\overline{w}\} > 0, (24) = 0 if\ Im\{\overline{w}\} = 0, (25) \leq |w|^{-2}\ Im\{\overline{w}\} if\ Im\{\overline{w}\} < 0. (26)$$

But the inequalities in (17) and (18) contradict, respectively, the inequalities in (8) and (9). Hence, we conclude that $|\Omega(z)| < 1$ for all $z \in U$. Consequently, it follows from (11) that

$$|(\frac{zf'(z)}{g(z)} - p)^w| = (p-\alpha)|\Omega(z)| < p - \alpha. \quad (27)$$

Therefore, the desired proof is completed. □

The Theorem 1 immediately yields the following interesting and important consequences.

Corollary 2. Let $f \in A_n(p), g \in S_n^*(p,\xi), \delta \in R$, $\{0\}, 0 \leq \alpha, \xi < p, p \in N, z \in U,$ and let the function H be defined by (7). Also, if $H(z)$ satisfies the following conditions:

$$Re\{H(z)\} = \{< \tfrac{1}{\delta} if\ \delta > 0, (28) > -\tfrac{1}{\delta} if\ \delta < 0, \text{or} Im\{H(z)\} \neq 0, (29)$$

then

$$Re\{\frac{zf'(z)}{g(z)}\} > p - (p-\alpha)^{1/\delta}. \quad (30)$$

Proof.. We choose ω as a real number and $w = \delta \in R$, $\{0\}$ in Theorem 1, then we obtain the corollary. □

Corollary 3. Let $f \in A_n(p), g \in S_n^*(p,\xi), 0 \leq \alpha, \xi < p, p \in N, z \in U,$ and let the function H be defined by (7). Also, if $H(z)$ satisfies the following conditions:

$$Re\{H(z)\} < 1 \text{or} Im\{H(z)\} \neq 0, \quad (31)$$

then $K_p(\alpha,\xi)$, that is, f is a p-valent close-to-convex function of order α and type ξ in U.

Proof.. Putting $w = 1$ in the Theorem 1, we can get the corollary. □

Corollary 4. Let $f \in A_n(p), 0 \leq \alpha < p, p \in N, z \in U,$ and let the function H be defined by

$$H(z) = (\frac{zf'(z)}{zf'(z) - pf(z)})(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}). \quad (32)$$

If $H(z)$ satisfies the following conditions:

$$Re\{H(z)\} < 1 \text{or} Im\{H(z)\} \neq 0, \quad (33)$$

then $f \in S_n^*(p,\alpha)$, that is, f is a p-valent starlike function of order α in U.

Proof.. Putting $g(z) = f(z)$ in the the corollary 3. □

Corollary 5. Let $f \in A_n(p), 0 \leq \alpha < p, p \in N, z \in U,$ and let the function H be defined by

$$H(z) = (\frac{f'(z)}{f'(z) - pz^{p-1}})(1 + \frac{zf''(z)}{f'(z)} - p). \quad (34)$$

If $H(z)$ satisfies the following conditions:

$$Re\{H(z)\} < 1 \text{or} Im\{H(z)\} \neq 0, \quad (35)$$

then $f \in C_n(p,\alpha)$, that is, f is a p-valent close-to-convex function of order α in U.

Proof.. Putting $g(z) = z^p$ in the the corollary 3. □

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