

Research Article

Trial Equation Method for Solving the (2+1)-Dimensional Kadomtsov-Petviashvili-Benjamin-Bona-Mahony Equation

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Abstract: Trial equation method is a powerful tool for obtaining exact solutions of nonlinear differential equations. In this paper, the Kadomtsov-Petviashvili-Benjamin-Bona-Mahony equation is reduced to an ordinary differential equation under the travelling wave transformation. Trial equation method and the theory of complete discrimination system for polynomial are used to establish exact solutions of the equation.

Keywords: the nonlinear partial differential equation; trial equation method; complete discrimination system for polynomial; traveling wave transform; the Kadomtsov-Petviashvili-Benjamin-Bona-Mahony equation.

INTRODUCTION

The nonlinear evolution equations (NLEEs) are widely used to describe physical phenomena in various fields of sciences, especially in fluid mechanics, solid state physics, plasma physics, plasma waves, biology and so on. It is significant to obtain their exact solutions. During the past few decades, various methods have been developed by researchers to find explicit solutions for the NLEEs.

The purpose of this article is to study the traveling wave solutions to the (2+1)-dimensional Kadomtsov-Petviashvili-Benjamin-Bona-Mahony (KP-BBM) equation by Liu's trial equation method [1-3] and the complete discrimination system for polynomial [4-7]. The equation is given by

$$(u_t + u_x - \alpha(u^2)_x - \beta u_{xt})_x + \gamma u_{yy} = 0. \quad (1)$$

Here, in Eq(1) α , β and γ are real valued constants.

The solutions of Eq(1) have been studied in various aspects. For example, Abdou [8] used the extended mapping method with symbolic computation to obtain some periodic solutions, solitary wave solution, and triangular wave solution of this equation, Wazwaz [9, 10] used the sine-cosine method, the tanh method and the extended tanh method for finding solitary solutions of it and so on.

DESCRIPTION OF TRIAL EQUATION METHOD

The objective of this section is to outline the use of trial equation method for solving a nonlinear partial differential equation (PDE). Suppose we have a nonlinear PDE for $u(x, y, t)$, in the form

$$P(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, u_{xt}, \dots) = 0 \quad (2)$$

where P is a polynomial, which includes nonlinear terms and the highest order derivatives and so on.

Step 1. Taking the wave transformation $u = u(\xi_1)$, $\xi_1 = k_1 x + k_2 y + \omega t$, reduces Eq. (2) to the ordinary differential equation (ODE).

$$M(u, ku', \omega u', k^2 u'', \omega^2 u'', k\omega u'', \dots) = 0 \quad (3)$$

Step 2. Take trial equation method

$$u'' = F(u) = a_0 + a_1 u + \dots + a_m u^m. \quad (4)$$

Integrating the Equation (4) with respect to ξ_1 once, we get

$$(u')^2 = H(u) = \frac{2a_m}{m+1} u^{m+1} + \dots + a_1 u^2 + 2a_0 u + d \tag{5}$$

where m, a_i and integration constant d are to be determined. Substituting Eqs. (4),(5) and other derivative terms into Eq.(3) yields a polynomial $G(u)$ of u . According to the balance principle we can determine the value of m . Setting the coefficients of $G(u)$ to zero, we get a system of algebraic equations. Solving this system, we can determine values of a_0, a_1, \dots, a_m and integration constant .

Step 3. Rewrite Eq.(5) by the integral form

$$\pm(\xi - \xi_0) = \int \frac{du}{\sqrt{H(u)}}. \tag{6}$$

According to the complete discrimination system of the polynomial, we classify the roots of $H(u)$ and solve the integral equation (6). Thus we obtain the exact traveling wave solutions to Eq.(2).

APPLICATION OF TRIAL EQUATION METHOD

Taking the traveling wave transformation $u = u(\xi_1)$ and $\xi_1 = k_1 x + k_2 y + \omega t$, we can obtain the corresponding reduced ODE of Eq(1).

$$(k_1^2 + k_1 \omega + \gamma k_2^2)u'' - 2\alpha k_1^2 (u')^2 - 2\alpha k_1^2 uu'' - \beta k_1^3 \omega u''' = 0. \tag{7}$$

we take the trial equation as follows

$$u'' = a_0 + a_1 u + \dots + a_m u^m. \tag{8}$$

According to the trial equation method of rank homogeneous equation, balancing u''' with uu'' (or $(u')^2$) gets $m = 2$, so Eq.(8) has the following specific form

$$u'' = a_0 + a_1 u + a_2 u^2. \tag{9}$$

Integrating Eq.(9) with respect to ξ_1 once, we yield

$$(u')^2 = \frac{2}{3} a_2 u^3 + a_1 u^2 + 2a_0 u + d. \tag{10}$$

where values of a_0, a_1, a_2 and the integration constant d are to be determined latter. By Eq.(9) and Eq.(10),we derive the following formula

$$u''' = \frac{10}{3} a_2^2 u^3 + 5a_1 a_2 u^2 + (6a_0 a_2 + a_1^2)u + a_0 a_1 + 2a_2 d. \tag{11}$$

Substituting Eqs.(9),(10) and (11) into Eq.(7),we have

$$h_3 u^3 + h_2 u^2 + h_1 u + h_0 = 0. \tag{12}$$

where

$$h_0 = (k_1^2 + k_1 \omega + \gamma k_2^2)a_0 - 2\alpha k_1^2 d - \beta k_1^3 \omega (a_0 a_1 + 2a_2 d). \tag{13}$$

$$h_1 = (k_1^2 + k_1 \omega + \gamma k_2^2)a_1 - 6\alpha k_1^2 a_0 - \beta k_1^3 \omega (a_1^2 + 6a_0 a_2). \tag{14}$$

$$h_2 = (k_1^2 + k_1 \omega + \gamma k_2^2)a_2 - 4\alpha k_1^2 a_1 - 5\beta k_1^3 \omega a_1 a_2. \tag{15}$$

$$h_3 = -\frac{10}{3} \alpha k_1^2 a_2 - \frac{10}{3} \beta k_1^3 \omega a_2^2. \tag{16}$$

Let the coefficient $h_i = 0(i = 0, 1, 2, 3)$ be zero, we will yield nonlinear algebraic equations. Solving the equations, we will determine the values of a_0, a_1, a_2, d . We get $a_1 = \frac{k_1^2 + k_1 \omega + \gamma k_2^2}{\beta k_1^3 \omega}, a_2 = -\frac{\alpha}{\beta k_1 \omega}, a_0$ and d are two arbitrary constants. When the above conditions are satisfied, we use the complete discrimination system for the third order polynomial and have the following solving process.

Let

$$v = (\frac{2}{3} a_2)^{\frac{1}{3}} u, \xi = (\frac{2}{3} a_2)^{\frac{1}{3}} \xi_1, d_2 = a_1 (\frac{2}{3} a_2)^{-\frac{2}{3}}, d_1 = 2a_0 (\frac{2}{3} a_2)^{-\frac{1}{3}}, d_0 = d. \tag{17}$$

Then Eq.(10) becomes

$$(v')^2 = v^3 + d_2v^2 + d_1v + d_0. \tag{18}$$

Where v is a function of ξ . The integral form of Eq.(18) is

$$\pm(\xi - \xi_0) = \int \frac{dv}{\sqrt{v^3 + d_2v^2 + d_1v + d_0}}. \tag{19}$$

Denote

$$F(v) = v^3 + d_2v^2 + d_1v + d_0. \tag{20}$$

$$\Delta = -27\left(\frac{2d_2^3}{27} + d_0 - \frac{d_1d_3}{3}\right)^2 - 4(d_1 - d_2^2)^3, D_1 = d_1 - \frac{d_2^2}{3}. \tag{21}$$

According to the complete discrimination system, we give the corresponding single traveling wave solutions to Eq.(6).

Case1. $\Delta = 0, D_1 < 0. F(v) = 0$ has a double real root and a simple real root. Then we have

$$F(v) = (v - \lambda_1)^2(v - \lambda_2), \lambda_1 \neq \lambda_2. \tag{22}$$

When $v > \lambda_2$, the corresponding solutions are

$$u_1 = \left(\frac{2}{3}a_2\right)^{-\frac{1}{3}}\{(\lambda_1 - \lambda_2) \tanh^2\left[\frac{\sqrt{\lambda_1 - \lambda_2}}{2}\left(\frac{2}{3}a_2\right)^{\frac{1}{3}}(k_1x + k_2y + \omega t - \xi_0)\right] + \lambda_2\}, (\lambda_1 > \lambda_2); \tag{23}$$

$$u_2 = \left(\frac{2}{3}a_2\right)^{-\frac{1}{3}}\{(\lambda_1 - \lambda_2) \coth^2\left[\frac{\sqrt{\lambda_1 - \lambda_2}}{2}\left(\frac{2}{3}a_2\right)^{\frac{1}{3}}(k_1x + k_2y + \omega t - \xi_0)\right] + \lambda_2\}, (\lambda_1 > \lambda_2); \tag{24}$$

$$u_3 = \left(\frac{2}{3}a_2\right)^{-\frac{1}{3}}\{(-\lambda_1 + \lambda_2) \sec^2\left[\frac{\sqrt{-\lambda_1 + \lambda_2}}{2}\left(\frac{2}{3}a_2\right)^{\frac{1}{3}}(k_1x + k_2y + \omega t - \xi_0)\right] + \lambda_1\}, (\lambda_1 < \lambda_2). \tag{25}$$

Case2. $\Delta = 0, D_1 = 0. F(v) = 0$ has a triple root. Then we have

$$F(v) = (v - \lambda)^3. \tag{26}$$

The corresponding solution is

$$u_4 = 4\left(\frac{2}{3}a_2\right)^{-\frac{2}{3}}(k_1x + k_2y + \omega t - \xi_0)^{-2} + \lambda. \tag{27}$$

Case 3. $\Delta > 0, D_1 < 0. F(v) = 0$ has three different real roots. Then we have

$$F(v) = (v - \lambda_1)(v - \lambda_2)(v - \lambda_3), \lambda_1 < \lambda_2 < \lambda_3. \tag{28}$$

When $\lambda_1 < v < \lambda_2$, we take the transformation as follows

$$v = \lambda_1 + (\lambda_2 - \lambda_1)\sin^2\phi. \tag{29}$$

According to the Eq.(19), we have

$$\pm(\xi - \xi_0) = \int \frac{dv}{\sqrt{F(v)}} = \frac{2}{\sqrt{\lambda_3 - \lambda_1}} \int \frac{d\phi}{\sqrt{1 - m^2 \sin^2\phi}}. \tag{30}$$

Where $m^2 = \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}$. On the basis of Eq.(30) and the definition of the Jacobi elliptic sine function, we have

$$v = \lambda_1 + (\lambda_2 - \lambda_1) \operatorname{sn}^2\left(\frac{\sqrt{\lambda_3 - \lambda_1}}{2}(\xi_1 - \xi_0), m\right). \tag{31}$$

The corresponding solutions is

$$u_5 = \left(\frac{2}{3}a_2\right)^{-\frac{1}{3}} \left[\lambda_1 + (\lambda_2 - \lambda_1) \operatorname{sn}^2\left(\frac{\sqrt{\lambda_3 - \lambda_1}}{2}\left(\frac{2}{3}a_2\right)^{\frac{1}{3}}(k_1x + k_2y + \omega t - \xi_0), m\right) \right]. \tag{32}$$

When $v > \lambda_3$, we take the transformation as follows

$$v = \frac{-\lambda_2 \sin^2\phi + \lambda_3}{\cos^2\phi}. \tag{33}$$

The corresponding solutions is

$$u_6 = \left(\frac{2}{3} a_2\right)^{-\frac{1}{3}} \left[\frac{\lambda_3 - \lambda_2 \operatorname{sn}^2\left(\frac{\sqrt{\lambda_3 - \lambda_1}}{2} \left(\frac{2}{3} a_2\right)^{\frac{1}{3}} (k_1 x + k_2 y + \omega t - \xi_0), m\right)}{\operatorname{cn}^2\left(\frac{\sqrt{\lambda_3 - \lambda_1}}{2} \left(\frac{2}{3} a_2\right)^{\frac{1}{3}} (k_1 x + k_2 y + \omega t - \xi_0), m\right)} \right]. \quad (34)$$

where $m^2 = \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}$.

Case4. $\Delta < 0, F(v) = 0$ has only a real root. Then we have

$$F(v) = (v - \lambda)(v^2 + pv + q), p^2 - 4q < 0. \quad (35)$$

When $v > \lambda_1$, we take the transformation as follows

$$v = \lambda + \sqrt{\lambda^2 + p\lambda + q} \tan^2 \frac{\phi}{2}. \quad (36)$$

According to the Eq.(19), we have

$$\xi - \xi_0 = \int \frac{dv}{\sqrt{(v - \lambda)(v^2 + pv + q)}} = \frac{1}{(\lambda^2 + p\lambda + q)^{\frac{1}{4}}} \int \frac{d\phi}{\sqrt{1 - m^2 \sin^2 \phi}}. \quad (37)$$

where $m^2 = \frac{1}{2} \left(1 - \frac{\lambda + \frac{p}{2}}{\sqrt{\lambda^2 + p\lambda + q}}\right)$. On the basis of Eq.(37) and the definition of the Jacobi elliptic cosine function, we have

$$v = \lambda + \frac{2\sqrt{\lambda^2 + p\lambda + q}}{1 + \operatorname{cn}\left((\lambda^2 + p\lambda + q)^{\frac{1}{4}} \left(\frac{2}{3} a_2\right)^{\frac{1}{3}} (k_1 x + k_2 y + \omega t - \xi_0), m\right)} - \sqrt{\lambda^2 + p\lambda + q}. \quad (38)$$

The corresponding solutions is

$$u_7 = \left(\frac{2}{3} a_2\right)^{-\frac{1}{3}} \left[\lambda + \frac{2\sqrt{\lambda^2 + p\lambda + q}}{1 + \operatorname{cn}\left((\lambda^2 + p\lambda + q)^{\frac{1}{4}} \left(\frac{2}{3} a_2\right)^{\frac{1}{3}} (k_1 x + k_2 y + \omega t - \xi_0), m\right)} - \sqrt{\lambda^2 + p\lambda + q} \right]. \quad (39)$$

In Eqs.(23)(24)(25)(27)(32)(34) and (39), the integration constant ξ_0 has been rewritten, but we still use it. The solutions $u_i (i = 1, \dots, 7)$ are all possible exact traveling wave solutions to Eq.(1). We can see it is easy to write the corresponding solutions to the KP-BBM equation .

CONCLUSION

Trial equation method is a systematic method to solve nonlinear differential equations. The advantage of this method is that we can deal with nonlinear equations with linear methods. This method has the characteristics of simple steps and clear effectivity. Based on the idea of the trial equation method and the aid of the computerized symbolic computation, some exact traveling wave solutions to the KP-BBM equation have been obtained. With the same method, some of other equations can be dealt with.

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