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Research Article

Existence results for a coupled system of fourth-order differential equations with two-point boundary conditions

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Abstract: This paper studies a coupled system of fourth-order differential equations with two-point boundary conditions. Applying the Schauder's fixed point theorem, an existence result is proved.
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INTRODUCTION

Many physical systems cannot be described by a single differential equation but in fact, are modeled by a system of coupled equations. So the study of propagation of signals in a system of electrical cables led to the investigation of a system of linear differential equations. Some results related to these systems have been obtained in [1-3] and others. Coupled systems of differential equations also appear in the study of temperature distribution in a composite heat conductor. In consequence, the subject of coupled systems is gaining much importance and attention. For detail, see [4,6] and the references therein. The aim of this paper is to find positive solutions of coupled systems of fourth-order differential equations of the type

$$\begin{cases} (p_1(t)u''')'+q_1(t)u''=f_1(t,v)+e_1(t), \\ (p_2(t)v''')'+q_2(t)v''=f_2(t,u)+e_2(t), \\ \alpha_1 u(0)-\beta_1 u'(0)=0,\ \gamma_1 u(1)+\delta_1 u'(1)=0, \\ \xi_1 u''(0)-\eta_1 u'''(0)=0,\ \zeta_1 u''(1)+\theta_1 u'''(1)=0, \\ \alpha_2 v(0)-\beta_2 v'(0)=0,\ \gamma_2 v(1)+\delta_2 v'(1)=0, \\ \xi_2 v''(0)-\eta_2 v'''(0)=0,\ \zeta_2 v''(1)+\theta_2 v'''(1)=0. \end{cases} \tag{1.1}$$

Throughout this paper, we always suppose that

(S_1) $p_i(t)\in C^1([0,1],R),\ p_i(t)>0,\ q_i(t)\in C([0,1],R),\ q_i(t)\le 0, e_i(t)\in C([0,1],R)$, $\alpha_i,\beta_i,\gamma_i$, $\delta_i\ge 0,\xi_i,\eta_i,\zeta_i,\theta_i\ge 0,$ *and* $\beta_i\gamma_i+\alpha_i\gamma_i+\alpha_i\delta_i>0,\eta_i\zeta_i+\xi_i\zeta_i+\xi_i\theta_i>0 (i=1,2).$ $f_1, f_2\in$ $C([0,1]\times(0,+\infty),(0,+\infty))$, and may be singular near the zero.

The remaining part of the paper is organized as follows. In Section 2,some preliminary results will be given. In Sections 3,by employing a basic application of Schauder's fixed point theorem, we state and prove the existence results for (1.1) under the non-negativeness of the Green's function associated with (2.2)-(2.3). Our viewpoints shed some new light on problems with weak force potentials

PRELIMINARY

First, we discuss the existence of positive solutions of fourth-order boundary value problem

$$\begin{cases} (p(t)u''')' + q(t)u'' = e(t), \\ \alpha u(0) - \beta u'(0) = 0, \ \gamma u(1) + \delta u'(1) = 0, \\ \xi u''(0) - \eta u'''(0) = 0, \ \zeta u''(1) + \theta u'''(1) = 0. \end{cases} \tag{2.1}$$

Let $Q = I \times I$ and $Q_1 = \{(t,s) \in Q \mid 0 \le t \le s \le 1\}, Q_2 = \{(t,s) \in Q \mid 0 \le s \le t \le 1\}$. We denote the Green's functions for the following boundary value problems

$$\begin{cases} -u''(t) = 0, \\ \alpha u(0) - \beta u'(0) = 0 \\ \gamma u(1) + \delta u'(1) = 0, \end{cases} \tag{2.2}$$

and

$$\begin{cases} -(p(t)u'(t))' - q(t)u(t) = 0, \\ \xi u(0) - \eta u'(0) = 0, \\ \zeta u(1) + \theta u'(1) = 0, \end{cases} \tag{2.3}$$

by $H(t,s)$ and $G(t,s)$,respectively. It is well known that $H(t,s)$ and $G(t,s)$ can be written by

$$H(x,y) := \frac{1}{\rho} \begin{cases} (\beta + \alpha t)(\delta + \gamma(1-s)), (t,s) \in Q_1, \\ (\beta + \alpha s)(\delta + \gamma(1-t)), (t,s) \in Q_2. \end{cases}$$

where $\rho = \beta\gamma + \alpha\gamma + \alpha\delta > 0$ and

$$G(t,s) := \frac{1}{\omega} \begin{cases} m(t)n(s), \quad (t,s) \in Q_1, \\ m(s)n(t), \quad (t,s) \in Q_2. \end{cases}$$

Lemma 2.1: Suppose that (S_1) holds, then the Green's function $G(t,s)$, possesses the following properties:

(i): $m(t) \in C^2(I,R)$ is increasing and $m(t) > 0, x \in (0,1]$.

(ii): $n(t) \in C^2(I,R)$ is decreasing and $n(t) > 0, x \in [0,1)$.

(iii): $(Lm)(t) \equiv 0, m(0) = \eta, m'(0) = \xi$.

(iv): $(Ln)(t) \equiv 0, n(1) = \theta, n'(1) = -\zeta$.

(v): ω is a positive constant. Moreover, $p(t)(m'(t)n(t) - m(t)n'(t)) \equiv \omega$.

(vi): $G(t,s)$ is continuous and symmetrical over Q.

(vii): $G(t,s)$ has continuously partial derivative over Q_1, Q_2.

(viii): For each fixed $s \in I$, $G(t,s)$ satisfies $LG(t,s) = 0$ for $s \ne t, t \in I$. Moreover, $R_1(G) = R_2(G) = 0$ for $s \in (0,1)$.

(viiii): G_t' has discontinuous point of the first kind at $t = s$ and

$$G_t'(s+0,s) - G_t'(s-0,s) = -\frac{1}{p(s)}, s \in (0,1).$$

Suppose that u is a positive solution of (2.1).Then

$$u(t) = \int_0^1 \int_0^1 H(t,\tau) G(\tau,s) e(s) ds d\tau \qquad 0 \le t \le 1,$$

We define the function $\gamma_i(t) : [0,1] \to R$ by

$$\gamma_i(t) = \int_0^1 \int_0^1 H_i(t,\tau) G_i(\tau,s) e_i(s) ds d\tau, \ i = 1,2,$$

which is the unique solution of

$$\begin{cases} (p_i(t)u'''(t))' + q_i(t)u''(t) = e_i(t), \quad i = 1,2 \\ \alpha_i u(0) - \beta_i u'(0) = 0, \gamma_i u(1) + \delta_i u'(1) = 0. \\ \xi_i u''(0) - \eta_i u'''(0) = 0, \ \zeta_i u''(1) + \theta_i u'''(1) = 0. \end{cases}$$

Following from Lemma 2.1 and (S_1), it is easy to see that

$$G_i(t,s) > 0, H_i(t,s) > 0 \quad for\ all\ (t,s) \in [0,1]\times[0,1], i = 1,2$$

Let us fix some notation to be used in the following: For a given function $h \in C[0,1]$,we denote the essential supremum and infimum by h^* *and* h_*, if they exist. Let, $\gamma_{i*} = \min_t \gamma_i(t),\ \ \gamma_i^* = \max_t \gamma_i(t),$

MAIN RESULTS

1) $\gamma_{1*} \geq 0, \gamma_{2*} \geq 0$

Theorem 3.1.We assume that there exists $b_i \geq 0, \hat{b}_i \geq 0$ *and* $0 < \alpha_i < 1$ such that

$$(H_1) \qquad \frac{\hat{b}_i(t)}{u^{\alpha_i}} \leq f_i(t,u) \leq \frac{b_i(t)}{u^{\alpha_i}},\ for\ all\ u > 0, a.e. t \in (0,1), i = 1,2$$

If $\gamma_{1*} \geq 0, \gamma_{2*} \geq 0$, then there exists a positive solution of (1.1)

Proof A positive solution of (1.1) is just a fixed point of the completely continuous map $A(u,v) = (A_1u, A_2v) : C[0,1]\times C[0,1] \to C[0,1]\times C[0,1]$ defined as

$$\begin{aligned}(A_1u)(t) &:= \int_0^1\int_0^1 H_1(t,\tau)G_1(\tau,s)[f_1(s,v(s)) + e_1(s)]dsd\tau \\ &= \int_0^1\int_0^1 H_1(t,\tau)G_1(\tau,s)f_1(s,v(s))dsd\tau + \gamma_1(t);\end{aligned}$$

$$\begin{aligned}(A_2v)(t) &:= \int_0^1\int_0^1 H_2(t,\tau)G_2(\tau,s)[f_2(s,u(s)) + e_2(s)]dsd\tau \\ &= \int_0^1\int_0^1 H_2(t,\tau)G_2(\tau,s)f_2(s,u(s))dsd\tau + \gamma_2(t);\end{aligned}$$

By a direct application of Schauder's fixed point theorem, the proof is finished if we prove that A maps the closed convex set defined as

$$K = \{(u,v) \in C[0,1]\times C[0,1] : r_1 \leq u(t) \leq R_1, r_2 \leq v(t) \leq R_2,\ for\ all\ t \in [0,1]\}$$

into itself, where $R_1 > r_1 > 0, R_2 > r_2 > 0$ are positive constants to be fixed properly. For convenience, we introduce the following notations

$$\beta_i(t) = \int_0^1\int_0^1 H_i(t,\tau)G_i(\tau,s)b_i(s)dsd\tau, \quad \hat{\beta}_i(t) = \int_0^1\int_0^1 H_i(t,\tau)G_i(\tau,s)\hat{b}_i(s)dsd\tau, \quad i = 1,2.$$

Given $(u,v) \in K$,by the nonnegative sign of G_i and $f_i, i = 1,2$ we have

$$\begin{aligned}(A_1u)(t) &= \int_0^1\int_0^1 H_1(t,\tau)G_1(\tau,s)f_1(s,v(s))dsd\tau + \gamma_1(t) \\ &\geq \int_0^1 H_1(t,\tau)G_1(\tau,s)\frac{\hat{b}_1(s)}{v^{\alpha_1}(s)}dsd\tau \\ &\geq \int_0^1 H_1(t,\tau)G_1(\tau,s)\frac{\hat{b}_1(s)}{R_2^{\alpha_1}}dsd\tau \\ &\geq \hat{\beta}_{1*}\frac{1}{R_2^{\alpha_1}}\end{aligned}$$

Note for every $(u,v) \in K$

$$
\begin{aligned}
(A_1u)(t) &= \int_0^1\int_0^1 H_1(t,\tau)G_1(\tau,s)\,f_1(s,v(s))dsd\tau + \gamma_1(t)\\
&\le \int_0^1\int_0^1 H_1(t,\tau)G_1(\tau,s)\,\frac{b_1(s)}{v^{\alpha_1}(s)}\,dsd\tau + \gamma_1^*\\
&\le \int_0^1\int_0^1 H_1(t,\tau)G_1(\tau,s)\frac{b_1(s)}{r_2^{\alpha_1}}dsd\tau + \gamma_1^*\\
&\le \beta_1^*\frac{1}{r_2^{\alpha_1}} + \gamma_1^*
\end{aligned}
$$

Similarly, by the same strategy, we have

$$
\begin{aligned}
(A_2v)(t) &= \int_0^1\int_0^1 H_2(t,\tau)G_2(\tau,s)\,f_2(s,u(s))dsd\tau + \gamma_2(t)\\
&\ge \int_0^1\int_0^1 H_2(t,\tau)G_2(\tau,s)\,\frac{\hat{b}_2(s)}{u^{\alpha_2}(s)}\,dsd\tau\\
&\ge \int_0^1\int_0^1 H_2(t,\tau)G_2(\tau,s)\,\frac{\hat{b}_2(s)}{R_1^{\alpha_2}}\,dsd\tau\\
&\ge \beta_{2*}\frac{1}{R_1^{\alpha_2}}
\end{aligned}
$$

$$
\begin{aligned}
(A_2v)(t) &= \int_0^1\int_0^1 H_2(t,\tau)G_2(\tau,s)\,f_2(s,u(s))dsd\tau + \gamma_2(t)\\
&\le \int_0^1\int_0^1 H_2(t,\tau)G_2(\tau,s)\,\frac{b_2(s)}{u^{\alpha_2}(s)}\,dsd\tau + \gamma_2^*\\
&\le \int_0^1\int_0^1 H_2(t,\tau)G_2(\tau,s)\frac{b_2(s)}{r_1^{\alpha_2}}dsd\tau + \gamma_2^*\\
&\le \beta_2^*\frac{1}{r_1^{\alpha_2}} + \gamma_2^*
\end{aligned}
$$

Thus $(A_1u, A_2v) \in K$ if r_1, r_2, R_1, R_2 are chosen so that

$$
\beta_{1*}\cdot\frac{1}{R_2^{\alpha_1}} \ge \gamma_1,\quad \beta_1^*\cdot\frac{1}{r_2^{\alpha_1}} + \gamma_1^* \le R_1
$$

$$
\beta_{2*}\cdot\frac{1}{R_1^{\alpha_2}} \ge \gamma_2,\quad \beta_2^*\cdot\frac{1}{r_1^{\alpha_2}} + \gamma_2^* \le R_1
$$

Note that $\beta_{i*} > 0, \beta_{i*} > 0$ and taking $R = R_1 = R_2, r = r_1 = r_2, r = \frac{1}{R}$, it is sufficient to find $R > 1$ such that

$$
\beta_{1*}\cdot R^{1-\alpha_1} \ge 1,\quad \beta_1^*\cdot R^{\alpha_1} + \gamma_1^* \le R
$$

$$
\beta_{2*}\cdot R^{1-\alpha_2} \ge 1,\quad \beta_2^*\cdot R^{\alpha_2} + \gamma_2^* \le R
$$

and these inequalities hold for R big enough because $\alpha_i < 1.$

2) $\gamma_1^* \le 0, \gamma_2^* \le 0$

The aim of this section is to show that the presence of a weak singular nonlinearity makes it possible to find positive solutions if $\gamma_1^* \le 0, \gamma_2^* \le 0$

Theorem 3.2. We assume that there exists $b_i \ge 0, \hat{b}_i \ge 0$ *and* $0 < \alpha_i < 1$ such that (H_1) is satisfied .If $\gamma_1^* \le 0, \gamma_2^* \le 0$ and

$$r_{1*} \geq [\alpha_1\alpha_2 \cdot \frac{\tilde{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}}]^{\frac{1}{1-\alpha_1\alpha_2}}(1-\frac{1}{\alpha_1\alpha_2}),$$

$$r_{2*} \geq [\alpha_1\alpha_2 \cdot \frac{\tilde{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}}]^{\frac{1}{1-\alpha_1\alpha_2}}(1-\frac{1}{\alpha_1\alpha_2}) \tag{3.1}$$

then there exists a positive solution of (1.1)

Proof In this case, to prove that $A: K \to K$ it is sufficient to find $0 < r_1 < R_1$, $0 < r_2 < R_2$ such that

$$\frac{\tilde{\beta}_1^*}{R_2^{\alpha_1}} + \gamma_{1*} \geq r_1,\ \frac{\beta_1^*}{r_2^{\alpha_1}} \leq R_1 \tag{3.2}$$

$$\frac{\tilde{\beta}_{2*}}{R_1^{\alpha_2}} + \gamma_{2*} \geq r_2,\ \frac{\beta_2^*}{r_1^{\alpha_2}} \leq R_2 \tag{3.3}$$

If we fix $R_1 = \frac{\beta_1^*}{r_2^{\alpha_1}}, R_2 = \frac{\beta_2^*}{r_1^{\alpha_2}}$, then the first inequality of (3.3) holds if r_2 satisfies

$$\tilde{\beta}_{2*}(\beta_1^*)^{-\alpha_2} r_2^{\alpha_1\alpha_2} + \gamma_{2*} \geq r_2$$

or equivalently

$$\gamma_{2*} \geq g(r_2) := r_2 - \frac{\tilde{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}} r_2^{\alpha_1\alpha_2}$$

The function $g(r_2)$ possesses a minimum at

$$r_{20} := [\alpha_1\alpha_2 \cdot \frac{\tilde{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}}]^{\frac{1}{1-\alpha_1\alpha_2}}$$

Taking $r_2 = r_{20}$,then (3.3) holds if

$$\gamma_{2*} \geq g(r_{20}) = [\alpha_1\alpha_2 \cdot \frac{\tilde{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}}]^{\frac{1}{1-\alpha_1\alpha_2}}(1-\frac{1}{\alpha_1\alpha_2})$$

Similarly ,

$$\gamma_{1*} \geq h(r_1) := r_1 - \frac{\tilde{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}} r_1^{\alpha_1\alpha_2}$$

$h(r_1)$ possesses a minimum at

$$r_{10} := [\alpha_1\alpha_2 \cdot \frac{\tilde{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}}]^{\frac{1}{1-\alpha_1\alpha_2}}$$

$$\gamma_{1*} \geq [\alpha_1\alpha_2 \cdot \frac{\tilde{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}}]^{\frac{1}{1-\alpha_1\alpha_2}}(1-\frac{1}{\alpha_1\alpha_2})$$

Taking $r_1 = r_{10}, r_2 = r_{20}$, then the first inequalities in (3.2) and (3.3) hold if $\gamma_{1*} \geq h(r_1)$ and $\gamma_{2*} \geq g(r_2)$, which are just condition (3.1).The second inequalities hold directly from the choice of R_1 and R_2, so it remains to prove that $R_1 = \frac{\beta_1^*}{r_{20}^{\alpha_1}} > r_{10}$, $R_2 = \frac{\beta_2^*}{r_{10}^{\alpha_2}} > r_{20}$ This is easily verified through elementary computations:

$$R_1 = \frac{\beta_1^*}{r_{20}^{\alpha_1}} = \frac{\beta_1^*}{\{[\alpha_1\alpha_2 \cdot \frac{\beta_{2*}}{(\beta_1^*)^{\alpha_2}}]^{\frac{1}{1-\alpha_1\alpha_2}}\}^{\alpha_1}}$$

$$= \frac{\beta_1^*}{[\alpha_1\alpha_2 \cdot \frac{\beta_{2*}}{(\beta_1^*)^{\alpha_2}}]^{\frac{\alpha_1}{1-\alpha_1\alpha_2}}} = \frac{(\beta_1^*)^{1+\frac{\alpha_1\alpha_2}{1-\alpha_1\alpha_2}}}{(\alpha_1\alpha_2 \cdot \beta_{2*})^{\frac{\alpha_1}{1-\alpha_1\alpha_2}}}$$

$$= \frac{(\beta_1^*)^{\frac{1}{1-\alpha_1\alpha_2}}}{[(\alpha_1\alpha_2 \cdot \beta_{2*})^{\alpha_1}]^{\frac{1}{1-\alpha_1\alpha_2}}} = [\frac{\beta_1^*}{(\alpha_1\alpha_2 \cdot \beta_{2*})^{\alpha_1}}]^{\frac{1}{1-\alpha_1\alpha_2}}$$

$$=[\frac{1}{(\alpha_1\alpha_2)^{\alpha_1}} \cdot \frac{\beta_1^*}{(\beta_{2*})^{\alpha_1}}]^{\frac{1}{1-\alpha_1\alpha_2}} > [\alpha_1\alpha_2 \cdot \frac{\beta_{1*}}{(\beta_2^*)^{\alpha_1}}]^{\frac{1}{1-\alpha_1\alpha_2}} = r_{10},$$

since $\beta_{i*} \le \beta_i^*, i = 1,2.$ Similarly, we have $R_2 > r_{20}$.

3) $\gamma_{1*} \ge 0, \gamma_2^* \le 0 (\gamma_1^* \le 0, \gamma_{2*} \ge 0)$

Theorem 3.3. Assume that (H_1) is satisfied .If $\gamma_{1*} \ge 0, \gamma_2^* \le 0$ and

$$\gamma_{2*} \ge r_{21} - \beta_{2*} \cdot \frac{r_{21}^{\alpha_1\alpha_2}}{(\beta_1^* + \gamma_1^* r_{21}^{\alpha_1})^{\alpha_2}} \tag{3.4}$$

where $0 < r_{21} < +\infty$ is a unique positive solution of equation

$$r_2^{1-\alpha_1\alpha_2}(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\beta_{2*} \tag{3.5}$$

then there exists a positive solution of (1.1).

Proof We follow the same strategy and notation as in the proof of ahead theorem. In this case,to prove that $A: K \to K,$ it is sufficient to find $r_1 < R_1, r_2 < R_2$ such that

$$\frac{\beta_{1*}}{R_2^{\alpha_1}} \ge r_1, \ \frac{\beta_2^*}{r_1^{\alpha_2}} \le R_2 \tag{3.6}$$

$$\frac{\beta_{2*}}{R_1^{\alpha_2}} + \gamma_{2*} \ge r_2, \ \frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^* \le R_1 \tag{3.7}$$

If we fix $R_2 = \frac{\beta_2^*}{r_1^{\alpha_2}},$ then the first inequality of (3.6) holds if r_1 satisfies

$$\frac{\beta_{1*}}{(\beta_2^*)^{\alpha_1}} \cdot r_1^{\alpha_1\alpha_2} \ge r_1, \tag{3.8}$$

or equivalently

$$0 < r_1 \le [\frac{\beta_{1*}}{(\beta_2^*)^{\alpha_1}}]^{\frac{1}{1-\alpha_1\alpha_2}} \tag{3.9}$$

If we chose $r_1 > 0$ small enough, then (3.9)holds, and R_2 is big enough.

If we fix $R_1 = \frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^*,$ then the first inequality of (3.7) holds if r_2 satisfies

$$\begin{aligned}\gamma_{2*} &\ge r_2 - \frac{\beta_{2*}}{R_1^{\alpha_2}} \\ &= r_2 - \beta_{2*} \cdot \frac{1}{(\frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^*)^{\alpha_2}} \\ &= r_2 - \beta_{2*} \cdot \frac{1}{(\frac{\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}}{r_2^{\alpha_1}})^{\alpha_2}} \\ &= r_2 - \beta_{2*} \cdot \frac{r_2^{\alpha_1\alpha_2}}{(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2}},\end{aligned}$$

or equivalently

$$\gamma_{2*} \ge f(r_2) := r_2 - \beta_{2*} \cdot \frac{r_2^{\alpha_1\alpha_2}}{(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2}} \tag{3.10}$$

According to

$$\begin{aligned} f'(r_2) &= 1 - \beta_{2*} \cdot \frac{1}{(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{2\alpha_2}} [\alpha_1\alpha_2 r_2^{\alpha_1\alpha_2 - 1}(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2} \\ &\quad - r_2^{\alpha_1\alpha_2}\alpha_2(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2 - 1}\alpha_1\gamma_1^* r_2^{\alpha_1 - 1}] \\ &= 1 - \frac{\beta_{2*}\alpha_1\alpha_2 r_2^{\alpha_1\alpha_2 - 1}}{(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2}}[1 - \frac{r_2^{\alpha_1}\gamma_1^*}{\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}}] \\ &= 1 - \alpha_1\alpha_2\beta_1^*\beta_{2*} r_2^{\alpha_1\alpha_2 - 1}(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{-1-\alpha_2},\end{aligned}$$

we have $f'(0) = -\infty, f'(+\infty) = 1$, then there exists r_{21} such that $f'(r_{21}) = 0$, and

$$\begin{aligned} f''(r_2) &= -[\alpha_1\alpha_2\beta_1^*\beta_{2*}(\alpha_1\alpha_2 - 1) r_2^{\alpha_1\alpha_2 - 2}(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{-1-\alpha_2} \\ &\quad + \alpha_1\alpha_2\beta_1^*\beta_{2*} r_2^{\alpha_1\alpha_2 - 1}(-1-\alpha_2)(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{-2-\alpha_2}\gamma_1^*\alpha_1 r_2^{\alpha_1 - 1}] > 0\end{aligned}$$

Then the function $f(r_2)$ possesses a minimum at $r_{21}, i.e., f(r_{21}) = \min_{r_2 \in (0,+\infty)} f(r_2)$.

Note $f'(r_{21}) = 0,$ then we have

$$1 - \alpha_1\alpha_2\beta_1^*\beta_{2*} r_{21}^{\alpha_1\alpha_2 - 1}(\beta_1^* + \gamma_1^* \cdot r_{21}^{\alpha_1})^{-1-\alpha_2} = 0$$

or equivalently

$$r_{21}^{1-\alpha_1\alpha_2}(\beta_1^* + \gamma_1^* \cdot r_{21}^{\alpha_1})^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\beta_{2*}$$

Taking $r_2 = r_{21},$ then the first inequality in (3.7)holds if $\gamma_{2*} \ge f(r_{21}),$ which is just condition (3.4). The second inequalities hold directly by the choice of R_1,and it would remain to prove that $r_{21} < R_2$ and $r_{10} < R_1$. These inequalities hold for R_2 big enough and r_1 small enough.

Remark 1. In theorem 3.3 the right-hand side of condition (3.4)always negative, this is equivalent to proof that $f(r_{21}) < 0.$ This is obviously established through the proof of Theorem 3.3.

Similarly, we have the following theorem.

Theorem 3.4.Assume (H_1) is satisfied .If $\gamma_1^* \le 0, \gamma_{2*} \ge 0$ and

$$\gamma_{1*} \ge r_{11} - \beta_{1*} \cdot \frac{r_{11}^{\alpha_1\alpha_2}}{(\beta_2^* + \gamma_2^* r_{11}^{\alpha_2})^{\alpha_1}},$$

where $0 < r_{11} < +\infty$ is a unique positive solution of the equation

$$r_1^{1-\alpha_1\alpha_2}(\beta_2^* + \gamma_2^* r_1^{\alpha_2})^{1+\alpha_1} = \alpha_1\alpha_2\beta_2^*\beta_{1*},$$

then there exists a positive solution of (1.1)

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