Research Article

The exact solutions to the Klein-Gordon-Zakharov equation

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Abstract: The complete discrimination system for polynomial method is applied to the the Klein-Gordon-Zakharov equation to obtain the classifications of the traveling wave solutions.

Keywords: traveling wave solution; complete discrimination system for polynomial method; the Klein-Gordon-Zakharov equation

INTRODUCTION

In this paper, we consider the Klein-Gordon-Zakharov equation[1].

$$\begin{cases} u_{tt} - u_{xx} + u + \alpha n u = 0, \\ n_{tt} - n_{xx} = \beta\left(|u|^2\right)_{xx}, \end{cases} \tag{1}$$

This system describes the interaction of the Langmuir wave and the ion acoustic wave in a high-frequency plasma. We use the complete discrimination system for polynomial method to Eq.(1) and give the classification of its all single traveling wave solutions. The complete discrimination system for polynomial method was proposed by Liu [2-4] in the past several years. By Liu's method, many nonlinear partial differential equations[5-7] have been solved.

CLASSIFICATION

Taking the following transformation to the Eq.(1)

$$u = u(\xi), \quad n = n(\xi), \quad \xi = x - ct , \tag{2}$$

then we can get the following equations

$$\begin{cases} c^2 u'' - u'' + u + \alpha n u = 0, \\ c^2 n'' - n'' = \beta\left(|u|^2\right)'', \end{cases} \tag{3}$$

Integrating Eq.(3), we have

$$u'' = \frac{1}{\left(c^2 - 1\right)^2}\left[-2\beta u^3 - 2c_1 u^2 - \left(2c_2 - c^2 + 1\right)u\right], \tag{4}$$

and

$$n = \frac{\beta u^2 + c_1 u + c_2}{c^2 - 1} . \tag{5}$$

Integrating Eq.(4), we get

$$\left(u'\right)^2 = a_4 u^4 + a_3 u^3 + a_2 u^2 + a_0 , \tag{6}$$

where $a_4 = -\dfrac{\beta}{2\left(c^2 - 1\right)^2}$, $a_3 = -\dfrac{2c_1}{3\left(c^2 - 1\right)^2}$, $a_2 = \dfrac{c^2 - c_2 - 1}{2\left(c^2 - 1\right)^2}$, a_0, c_1, c_2 are integration constants.

When $a_4 > 0$, we take the transformation $w = \left(a_4\right)^{\frac{1}{4}}\left(u + \dfrac{a_3}{4a_4}\right)$, $\xi_1 = \left(a_4\right)^{\frac{1}{4}}\xi$, $\tag{7}$

$$\text{Eq.(6) becomes} \qquad w_{\xi_1}^2 = w^4 + pw^2 + qw + r, \qquad (8)$$

$$\text{where } p = \frac{a_2}{\sqrt{a_4}}, \ q = \left(\frac{a_3^3}{8a_4^2} - \frac{a_2 a_3}{2a_4} \right) a_4^{-\frac{1}{4}}, \ r = \frac{-3a_3^4}{256a_4^3} + \frac{a_2 a_3^2}{16a_4^2} + a_0.$$

When $a_4 < 0$, we take the transformation $\qquad w = (-a_4)^{\frac{1}{4}}\left(u + \frac{a_3}{4a_4} \right), \quad \xi_1 = (-a_4)^{\frac{1}{4}}\xi, \qquad (9)$

$$\text{Eq.(6) becomes} \qquad w_{\xi_1}^2 = -(w^4 + pw^2 + qw + r), \qquad (10)$$

$$\text{where } p = \frac{-a_2}{\sqrt{-a_4}}, \ q = \left(-\frac{a_3^3}{8a_4^2} + \frac{a_2 a_3}{2a_4} \right)(-a_4)^{-\frac{1}{4}}, \ r = \frac{3a_3^4}{256a_4^3} - \frac{a_2 a_3^2}{16a_4^2} - a_0.$$

$$\text{Then Eq.(8) and Eq.(10) become} \quad w_{\xi_1}^2 = \varepsilon \ (w^4 + pw^2 + qw + r), \qquad (11)$$

where $\varepsilon = \pm 1$.

$$\text{Integrating Eq.(10) yields} \qquad \pm(\xi - \xi_0) = \int \frac{dw}{\varepsilon F(w)}, \qquad (12)$$

where $F(w) = w^4 + pw^2 + qw + r$.

Write the discrimination of $F(w)$ as follows

$$\begin{cases} D_1 = 4 \\ D_2 = -p \\ D_3 = -2p^3 + 8pr - 9q^2 \\ D_4 = -p^3 q^2 + 4p^4 r + 36pq^2 r - 32p^2 r^2 - \dfrac{27}{4}q^4 + 64r^3 \\ D_5 = 9p^2 - 32pq \end{cases} \qquad (13)$$

According to the discrimination , we give the corresponding traveling wave solutions to Eq.(12).

Case 2.1: $D_4 = 0, \ D_3 = 0, \ D_2 < 0$. we have $F(w) = \left[(w-l)^2 + s^2\right]^2$, where l, s are real numbers, $s > 0$.

When $\varepsilon = 1$, the corresponding traveling wave solutions of Eq.(12) are

$$w = s\tan[s(\xi - \xi_0)] + l \qquad (14)$$

Case 2.2: $D_4 = 0, \ D_3 = 0, \ D_2 = 0$. When $\varepsilon = 1$, we assume $F(w) = w^4$, the corresponding traveling wave solutions of Eq.(12) are

$$w = -(\xi - \xi_0)^{-1} \qquad (15)$$

Case 2.3: $D_4 = 0, \ D_3 = 0, \ D_2 > 0, E_2 = 0$. We assume $F(w) = (w - \alpha)^2(w - \beta)^2$, where α, β are real numbers, $\alpha > \beta$. If $\varepsilon = 1$, when $w > \alpha$ or $w < \beta$, the corresponding traveling wave solutions of Eq.(12) are

$$w = \frac{\beta - \alpha}{2}\left[\coth\frac{(\alpha - \beta)(\xi - \xi_0)}{2} - 1 \right] + \beta \qquad (16)$$

when $\beta < w < \alpha$, the corresponding traveling wave solutions of Eq.(12) are

$$w = \frac{\beta - \alpha}{2}\left[\tanh\frac{(\alpha - \beta)(\xi - \xi_0)}{2} - 1 \right] + \beta \qquad (17)$$

Case 2.4: $D_4 = 0, \ D_3 > 0, \ D_2 > 0$. We assume $F(w) = (w - \alpha)^2(w - \beta)(w - \gamma)$, $\beta > \gamma$

If $\varepsilon = 1$, when $\alpha > \beta, w > \beta$, or $\alpha < \gamma, w < \gamma$, the corresponding traveling wave solutions of Eq.(12) are

$$\pm(\xi - \xi_0) = \frac{1}{(\alpha - \beta)(\alpha - \gamma)}\ln\frac{\left[\sqrt{(w - \beta)(\alpha - \gamma)} - \sqrt{(\alpha - \beta)(w - \gamma)}\right]^2}{|w - \alpha|}; \qquad (18)$$

When $\alpha > \beta,\ w < \gamma$, or $\alpha < \gamma,\ w < \beta$, the corresponding traveling wave solutions of Eq.(12) are

$$\pm(\xi - \xi_0) = \frac{1}{(\alpha - \beta)(\alpha - \gamma)} \ln \frac{\left[\sqrt{(w-\beta)(\gamma-\alpha)} - \sqrt{(\beta-\alpha)(w-\gamma)}\right]^2}{|w-\alpha|} ; \qquad (19)$$

When $\gamma < \alpha < \beta$, the corresponding traveling wave solutions of Eq.(12) are

$$\pm(\xi - \xi_0) = \frac{1}{(\beta - \alpha)(\alpha - \gamma)} \arcsin \frac{(w-\beta)(\alpha-\gamma) - (\alpha-\beta)(w-\gamma)}{|(w-\alpha)(\beta-\gamma)|} ; \qquad (20)$$

If $\varepsilon = -1$, when $\alpha > \beta,\ w > \beta$, or $\alpha < \gamma,\ w < \gamma$, the corresponding traveling wave solutions of Eq.(12) are

$$\pm(\xi - \xi_0) = \frac{1}{(\beta - \alpha)(\alpha - \gamma)} \ln \frac{\left[\sqrt{(-w+\beta)(\alpha-\gamma)} - \sqrt{(\beta-\alpha)(w-\gamma)}\right]^2}{|w-\alpha|} ; \qquad (21)$$

When $\alpha > \beta,\ w < \gamma$, or $\alpha < \gamma,\ w < \beta$, the corresponding traveling wave solutions of Eq.(12) are

$$\pm(\xi - \xi_0) = \frac{1}{(\beta - \alpha)(\alpha - \gamma)} \ln \frac{\left[\sqrt{(-w+\beta)(\gamma-\alpha)} - \sqrt{(\alpha-\beta)(w-\gamma)}\right]^2}{|w-\alpha|} ; \qquad (22)$$

When $\gamma < \alpha < \beta$, the corresponding traveling wave solutions of Eq.(12) are

$$\pm(\xi - \xi_0) = \frac{1}{(\alpha - \beta)(\alpha - \gamma)} \arcsin \frac{(-w+\beta)(\alpha-\gamma) + (\beta-\alpha)(w-\gamma)}{|(w-\alpha)(\beta-\gamma)|}. \qquad (23)$$

Case 2.5: $D_4 = 0,\ D_3 = 0,\ D_2 > 0,\ E_2 = 0$. We assume $F(w) = (w-\alpha)^3(w-\beta)$, where α, β are real numbers. If $\varepsilon = 1$, when $w > \alpha, w > \beta$, or $w < \alpha, w < \beta$, the corresponding traveling wave solutions of Eq.(12) are

$$w = \frac{4(\alpha - \beta)}{(\beta - \alpha)^2(\xi - \xi_0)^2 - 4} + \alpha ; \qquad (24)$$

If $\varepsilon = 1$, when $w > \alpha, w < \beta$, or $w < \alpha, w > \beta$, the corresponding traveling wave solutions of Eq.(12) are

$$w = \frac{4(\alpha - \beta)}{-(\beta - \alpha)^2(\xi - \xi_0)^2 - 4} + \alpha . \qquad (25)$$

Case 2.6: $D_4 = 0, D_2 D_3 < 0$, we have $F(w) = (w-\alpha)^2[(w-l)^2 + s^2]$, where α is real number. If $\varepsilon = 1$, the corresponding traveling wave solutions of Eq.(12) are

$$w = \frac{[e^{\pm\sqrt{(\alpha-l)^2+s^2}\,(\xi-\xi_0)} - \dfrac{\alpha - 2l}{\sqrt{(\alpha-l)^2+s^2}}] + \sqrt{(\alpha-l)^2+s^2}\,(2-\gamma)}{[e^{\pm\sqrt{(\alpha-l)^2+s^2}\,(\xi-\xi_0)} - \dfrac{\alpha - 2l}{\sqrt{(\alpha-l)^2+s^2}}]^2 - 1} \qquad (26)$$

Case 2.7: $D_4 > 0, D_3 > 0, D_2 > 0$. We assume $F(w) = (w-\alpha_1)(w-\alpha_2)(w-\alpha_3)(w-\alpha_4)$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are real numbers, $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$.

Case 2.8: $D_4 < 0$, $D_2 D_3 \geq 0$. We assume $F(w) = (w-\alpha)(w-\beta)[(w-l)^2 + s^2]$, where $\alpha > \beta,\ l,\ s > 0$, and they are all real numbers.

Case 2.9: $D_4 > 0, D_2 D_3 \leq 0$. We assume $F(w) = [(w-l_1)^2 + s_1^2][(w-l_2)^2 + s_2^2]$, where α_1, α_2, l_1 and s_1 are real numbers, $s_1 > s_2 > 0$.

The corresponding traveling wave solutions of Eq.(12) in Case 2.7-Case 2.9 are the forms in Elliptic function, its form is compiles, we omit.

The solutions (14)- (26) are the classification of exact solutions to the Eq.(12), then by Eq.(5), Eq.(7) and Eq.(9), we can get the $u(x,t)$ and $v(x,t)$, they are the solutions of the Klein-Gordon-Zakharov equation.

CONCLUSION
In the present paper, we use the complete discrimination system for polynomial method to the Klein-Gordon-Zakharov equation, and we obtain the classification of envelop traveling wave solutions. These solutions contain trigonometric functions , rational functions, solitary wave solutions and so on.

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