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The Hochschild Cohomology Group of a Class of Directed Tree-Path Algebras

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Abstract **Review Article**

The Hochschild cohomology of associative algebra is closely related to its algebraic structure. According to the characteristics of double modules in directed tree-path algebra and Hochschild's theory, the Hochschild cohomology groups of some finite dimensional algebras have been studied deeply. In this paper, we calculate the Hochschild cohomology groups of a directed tree-path algebra with and without branches.

Keywords: Hochschild cohomology group; Path algebras; Directed graph.

INTRODUCTION

The homology group theory of associative algebra is very rich [1, 2], before the homology group theory of associative algebra was defined, people only studied some special theories of derivatives and extensions. Since Hochschild proposed the Hochschild group theory of finite dimensional associative algebra in 1945, it becomes a meaningful subject to use the Hochschild group theory to study finite dimensional algebra [3-5]. The low order Hochschild cohomology groups are closely related to the algebraic structure. Therefore, the calculation of Hochschild cohomology groups of various algebras is of great significance in algebraic representation theory. This paper mainly studies a class of Hochschild cohomology groups of directed tree-path algebra.

Preparative knowledge

Definition 1 [6] Assume A_i, A, B, C are $R-$modules, $i \in I$, then the following $R-$modules are isomorphic:

$$A \otimes_R B \cong B \otimes_R A \qquad (1)$$

$$R \otimes_R A \cong A \qquad (2)$$

$$\left(\bigoplus_{i\in I} A_i\right) \otimes_R B \cong \bigoplus_{i\in I}\left(A_i \otimes_R B\right) \qquad (3)$$

$$(A \otimes_R B) \otimes_R C \cong A \otimes_R (B \otimes_R C) \qquad (4)$$

Definition 2 [7] Assume k is a domain, Q is a directed graph, $A = kQ$ is a $k-$vector space based on the path of Q. For $p = \alpha_1 \cdots \alpha_m$ and $q = \beta_1 \cdots \beta_n$, define multiplication:

$$pq = \begin{cases} \alpha_1 \cdots \alpha_m \beta_1 \cdots \beta_n, & t(p) = s(q) \\ 0, & t(p) \neq s(q) \end{cases}$$

In this case, $A = kQ$ is an $k-$algebra, we called it path algebra of Q, path algebra for short.

Definition 3 [8] Assume A is a finite dimensional k −algebra, M is a finite dimensional A-A- bimodule.
$C=(C^i,d^i)(i\in Z)$

$$C=\cdots\to C^{-1}\xrightarrow{d^{-1}}C^0\xrightarrow{d^0}C^1\to\cdots\to C^i\xrightarrow{d^i}C^{i+1}\to\cdots$$

Is called Hochschild complex, where $C^i=0, d^i=0, \forall i<0, C^0=M, C^i=\mathrm{Hom}_k(A^{\otimes i},M), \forall i>0$.
$A^{\otimes i}$ Represents that A makes i-th tensor product with itself on the field k.

$$d^0m(a)=am-ma, \forall m\in M, a\in A$$

$$(d^if)(a_1\otimes\cdots\otimes a_{i+1})=a_1f(a_2\otimes\cdots\otimes a_{i+1})$$

$$+\sum_{j=1}^{i}(-1)^j f(a_1\otimes\cdots\otimes a_ja_{j+1}\otimes\cdots\otimes a_{i+1})+(-1)^{i+1}f(a_1\otimes\cdots\otimes a_i)a_{i+1}$$

With $f\in\mathrm{Hom}_k(A^{\otimes i},M)$. Denote $H^i(A,M)=H^i(C)=Kerd^i/\mathrm{Im}d^{i-1}, \forall i\in Z$, it is called the i-th Hochschild cohomology group of the coefficient of A in M.

In particular, when we take $M=A, H^i(A)=H^i(A,A)$ is called the i-th Hochschild cohomology group of algebra A.

The Hochschild Cohomology Group of a Directed Tree-path Algebras without branches

Proposition 2.1 Assume D is a finite dimensional directed tree graph without branches, $D_0=\{e_0,e_1\}$ is a Vertex set, $D_1=\{a_1\}$ is a set of directed edges with length 1, C is a Path algebra of D over K, then $H^0(C)\cong K\oplus K$, $H^1(C)=0$.

Proof According to the known conditions, the directed tree graph is $e_0\xrightarrow{a_1}e_1$.
For $\forall x\in Kerd_1$, $d_1(x)=0$. Assume $x=K_1e_0+K_2e_1+K_3a_1$, we have
$d_1(K_1e_0+K_2e_1+K_3a_1)=K_1\Delta(e_0)+K_2\Delta(e_1)+K_3\Delta(a_1)=K_3(e_0\otimes e_1)=0$,

Therefore $K_3=0$, so $x=K_1e_0+K_2e_1$, Hence
$H^0(C)\cong K\oplus K$.

In the following, we calculate $H^1(C)$.
For $\forall x\in Ker(d_2)$, $d_2(x)=0$. Let $x=K_1e_0\otimes a_1+K_2e_0\otimes e_1$, according to $\Delta(a_i)=e_{i-1}\otimes e_i$, $\Delta e_i=0$, we can derive that

$$\begin{aligned}d_2(x)&=(\Delta\otimes I-I\otimes\Delta)x\\&=K_1\Delta(e_0)\otimes a_1+K_2\Delta(e_0)\otimes e_1+K_3\Delta(a_1)\otimes e_1\\&\quad-K_1e_1\otimes\Delta(a_1)-K_2e_0\otimes\Delta(e_1)-K_3a_1\otimes\Delta(e_1)\\&=K_3e_0\otimes e_1\otimes e_1-K_1e_1\otimes e_0\otimes e_1=0\end{aligned}$$

Therefore $K_1=K_3=0$, so $x=K_2(e_0\otimes e_1)=K_2\Delta(a_2)\in\mathrm{Im}d_1$, Hence $Kerd_2\subset\mathrm{Im}d_1$. And because $d_2d_1=0$, we can obtain $\mathrm{Im}d_1\subset Kerd_2$, so $Kerd_2=\mathrm{Im}d_1$, hence $H^1(C)=0$.

Proposition 2.2 Assume D is a finite dimensional directed tree graph without branches, $D_0=\{e_0,e_1,e_2\}$ is a Vertex set, $D_1=\{a_1,a_2\}$ is a set of directed edges with length 1, C is a Path algebra of D over K, then
$H^0(C)\cong K\oplus K\oplus K$, $H^1(C)=0$.

Proof According to the known conditions, the directed tree graph is $e_0\xrightarrow{a_1}e_1\xrightarrow{a_2}e_2$.

For $\forall x \in Kerd_1$, $d_1(x)=0$. Let $x = K_1e_0 + K_2e_1 + K_3e_2 + K_4a_1 + K_5a_2 + K_6a_1a_2$, we have
$d_1(x) = \Delta(x) = K_1\Delta(e_0) + K_2\Delta(e_1) + K_3\Delta(e_2) + K_4\Delta(a_1) + K_5\Delta(a_2) + K_6\Delta(a_1a_2)$
$= K_4(e_0 \otimes e_1) + K_5(e_1 \otimes e_2) + K_6(a_1 \otimes e_2 + e_0 \otimes a_2) = 0$,
Then $K_4 = K_5 = K_6 = 0$, therefore $x = K_1e_0 + K_2e_1 + K_3e_2$, hence $H^0(C) \cong K \oplus K \oplus K$.

In the following, we calculate $H^1(C)$.
For $\forall x \in Ker(d_2)$, let

$$x = K_1(e_0 \otimes a_1) + K_2(e_0 \otimes e_1) + K_3(e_0 \otimes a_2) + K_4(e_0 \otimes e_2) + K_5(a_1 \otimes e_1)$$
$$+ K_6(a_1 \otimes a_2) + K_7(a_1 \otimes e_2) + K_8(e_1 \otimes a_2) + K_9(e_1 \otimes e_2) + K_{10}(a_2 \otimes e_2),$$

According to $d_2(x) = (\Delta \otimes I - I \otimes \Delta)x = 0$, we obtain
$K_1 = K_4 = K_5 = K_6 = K_8 = K_{10} = 0$, $K_3 = K_7$, K_2, K_9 are free variables.

Then we have
$x = K_3(e_0 \otimes a_2 + a_1 \otimes e_2) + K_2(e_0 \otimes e_1) + K_9(e_1 \otimes e_2)$
$= K_3\Delta(a_1a_2) + K_2\Delta(a_1) + K_9\Delta(a_2) \in \mathrm{Im}\, d_1$
Therefor $Kerd_2 \subset \mathrm{Im}\, d_1$. And because $d_2d_1 = 0$, thus $\mathrm{Im}\, d_1 \subset Kerd_2$, we derive that $Kerd_2 = \mathrm{Im}\, d_1$, hence $H^1(C) = 0$.

Proposition 2.3 Assume D is a finite dimensional directed tree graph without branches, $D_0 = \{e_0, e_1, e_2, \cdots, e_n\}$ is a Vertex set, $D = \{a_1, a_2, \cdots, a_n\}$ is a set of directed edges with length 1, C is a Path algebra of D over K ,then

$$H^0(C) \cong \overbrace{K \oplus K \oplus \cdots \oplus K}^{n+1个}, H^1(C) = 0$$

Proof By $e_0 \xrightarrow{a_1} e_1 \xrightarrow{a_2} e_2 \xrightarrow{a_3} \cdots \longrightarrow e_{n-1} \xrightarrow{a_n} e_n$, we structure complex
$C \xrightarrow{\Delta = d_1} C \otimes C \xrightarrow{d_2} C \otimes C \otimes C \xrightarrow{d_3} \cdots \xrightarrow{d_n} C^{\otimes n+1}$.
For $\forall x \in Kerd_1$, let
$x = K_1e_0 + K_2e_1 + \cdots + K_{n+1}e_n + l_1a_1 + \cdots + l_na_n + \sum_{i \in N^+} t_ia_1a_2\cdots a_i$,
By $d(x) = \Delta(x) = 0$, we obtain $l_i = 0 (i = 1, 2, \cdots, n)$, $t_i = 0 (i \in N^+)$, thus
$x = K_1e_0 + K_2e_1 + \cdots + K_{n+1}e_n$
So $H^0(C) \cong \overbrace{K \oplus K \oplus \cdots \oplus K}^{n+1个}$.

In the following, we prove $H^1(C) = 0$.
For $\forall x \in Ker(d_2)$, we have $d_2(x) = 0$. Because

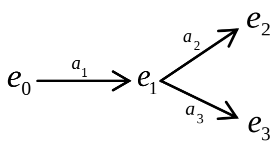
$$d_2(x) = (\Delta \otimes I - I \otimes \Delta)x$$
$$= (\Delta \otimes I - I \otimes \Delta)\left(\sum K_{j_1}(e_{i_1} \otimes a_{t_1}) + \sum K_{j_2}(a_{i_2} \otimes e_{t_2}) + \sum K_{j_3}(e_{i_3} \otimes e_{t_3})\right) = 0$$

Therefore
$x = \sum K_{j_2}(\Delta(a_{i_2}) \otimes e_{t_2}) - \sum K_{j_1}(e_{i_1} \otimes \Delta(a_{t_1}))$
$= \sum K_t\Delta(a_1a_2 + a_1a_3 + \cdots + a_1a_n + a_1a_2a_3 + \cdots + a_1a_2a_3\cdots a_n) \in \mathrm{Im}\, d_1$
We obtain $Kerd_2 \subset \mathrm{Im}\, d_1$, and because $d_2d_1 = 0$, so $\mathrm{Im}\, d_1 \subset Kerd_2$, hence $H^1(C) = 0$.

The Hochschild Cohomology Group of a Directed Tree-path Algebras with branches

Proposition 3.1 Assume D is a finite dimensional directed tree graph with branches, $D_0 = \{e_0, e_1, e_2, e_3\}$ is a Vertex set, $D_1 = \{a_1, a_2, a_3\}$ is a set of directed edges with length 1, C is a Path algebra of D over K, then $H^0(C) \cong K \oplus K \oplus K$, $H^1(C) = 0$.

Proof According to the known conditions, the directed tree graph is



Let $x \in Ker(d_1)$, then $d_1(x) = 0$. Assume

$$x = K_1e_0 + K_2e_1 + K_3e_2 + K_4e_3 + K_5a_1 + K_6a_2 + K_7a_3 + K_8a_1a_2 + K_9a_1a_3$$

According to

$$d_1(x) = \Delta(x) = K_5\Delta(a_1) + K_6\Delta(a_2) + K_7\Delta(a_3) + K_8\Delta(a_1a_2) + K_9\Delta(a_1a_3) = 0$$

We obtain $K_5 = K_6 = K_7 = K_8 = K_9 = 0$, so $x = K_1e_0 + K_2e_1 + K_3e_2 + K_4e_3$, hence we derive that $H^0(C) \cong K \oplus K \oplus K$.

In the following, we calculate $H^1(C)$.

For $\forall x \in Ker(d_2)$, we have $d_2(x) = 0$.

$$\begin{aligned} x = {} & K_1(e_0 \otimes a_1) + K_2(e_0 \otimes e_1) + K_3(e_0 \otimes a_2) + K_4(e_0 \otimes e_2) + K_5(a_1 \otimes e_1) \\ & + K_6(a_1 \otimes a_2) + K_7(a_1 \otimes e_2) + K_8(e_1 \otimes a_2) + K_9(e_1 \otimes e_2) + K_{10}(a_2 \otimes e_2) \\ & + K_{11}(e_0 \otimes a_3) + K_{12}(e_0 \otimes e_3) + K_{13}(a_1 \otimes e_1) + K_{14}(a_1 \otimes a_3) + K_{15}(a_1 \otimes e_3) \\ & + K_{16}(e_1 \otimes a_3) + K_{17}(e_1 \otimes e_3) + K_{18}(a_3 \otimes a_3) \\ & + K_6(a_1 \otimes a_2) + K_7(a_1 \otimes e_2) + K_8(e_1 \otimes a_2) + K_9(e_1 \otimes e_2) + K_{10}(a_2 \otimes e_2) \end{aligned}$$

By $d_2(x) = (\Delta \otimes I - I \otimes \Delta)x = 0$, we obtain $K_{11} = K_{15}, K_3 = K_7$, K_2, K_9, K_{17} are free variables, the others are both zero. Then

$$\begin{aligned} x = {} & K_2(e_0 \otimes e_1) + K_3(e_0 \otimes a_2 + a_1 \otimes e_2) + K_9(e_1 \otimes e_2) \\ & + K_{11}(e_0 \otimes a_3 + a_1 \otimes e_3) + K_{17}(e_1 \otimes e_3) \end{aligned}$$

Thus we obtain $Kerd_2 \subset \operatorname{Im} d_1$, and because $d_2d_1 = 0$, so $\operatorname{Im} d_1 \subset Kerd_2$, hence $H^1(C) = 0$.

Proposition 3.2 Assume D is a finite dimensional directed tree graph with branches, $D_0 = \{e_0, e_1, \cdots, e_n\}$ is a Vertex set, $D_1 = \{a_1, a_2, \cdots, a_n\}$ is a set of directed edges with length 1, C is a Path algebra of D over K, then

$$H^0(C) \cong \overbrace{K \oplus K \oplus \cdots \oplus K}^{n+1\text{个}}, H^1(C) = 0.$$

Proof Firstly we calculate $H^0(C)$.

For $\forall x \in Kerd_1$, let $x = K_1e_0 + K_2e_1 + \cdots + K_{n+1}e_n + \sum K_l a_{i1}a_{i2}\cdots a_{in}$, by $\Delta(x) = 0$, we obtain $K_l = 0$.

Thus

$$x = K_1e_0 + K_2e_1 + \cdots + K_{n+1}e_n$$

So $H^0(C) \cong \overbrace{K \oplus K \oplus \cdots \oplus K}^{n+1\text{个}}$.

In the following, we calculate $H^1(C)$.

For $\forall x \in Ker(d_2)$, we have $d_2(x) = 0$. let

$$x = \sum K_{i_1}(e_{j_1} \otimes a_{t_1}) + \sum K_{i_2}(a_{j_2} \otimes e_{t_2}) + \sum K_{i_3}(e_{j_3} \otimes e_{t_3})$$

By computing

$$d_2(x) = (\Delta \otimes I - I \otimes \Delta)x$$
$$= \sum K_{i_2}(\Delta(a_{j_2}) \otimes e_{t_2}) - \sum K_{i_1}(e_{j_1} \otimes \Delta(a_{t_1})) = 0$$

We obtain $x = \sum L\Delta(a_1 a_2 a_3 \cdots a_n) \subset \operatorname{Im} d_1$, therefor $Kerd_2 \subset \operatorname{Im} d_1$. And because $d_2 d_1 = 0$, we have $\operatorname{Im} d_1 \subset Kerd_2$, hence $H^1(C) = 0$.

CONCLUSIONS

In this paper, the Hochschild cohomology groups of a directed tree-path algebra with and without branches are calculated separately. We conclude that the zero-order Hochschild Homology results of directed tree-path algebra are not related to branches, but are related to the vertex set D_0 of the directed graph. Simultaneously we derive that the First Cohomology Group is trivial.

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