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Observation on the Binary Quadratic Equation $3x^2-8xy+3y^2+2x+2y+6=0$

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Abstract: The binary quadratic equation $3x^2-8xy+3y^2+2x+2y+6=0$ is studied for its non-trivial integral solutions. The recurrence relations satisfied by the solutions x and y are given. A few interesting properties among the solutions are presented.
Keywords: Binary quadratic equation, Integral solutions.
MSC subject classification: 11D09.

INTRODUCTION

The binary quadratic Diophantine equations (both homogeneous and non homogeneous) are rich in variety [1-6]. In [7-14] the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions.

However, in [13] it is shown that the hyperbola represented by $3x^2+xy=14$ has only finite number of integral points. These results have motivated us to search for infinitely many non-zero integral solutions of yet another interesting binary quadratic equation given by $3x^2-8xy+3y^2+2x+2y+6=0$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

METHOD OF ANALYSIS

The Diophantine equation under consideration is

$$3x^2-8xy+3y^2+2x+2y+6=0 \tag{1}$$

It is to be noted that (1) represents a hyperbola. By shifting the origin to the centre (1,1), (1) reduces to

$$3X^2-8XY+3Y^2=-8 \tag{2}$$

where $x=X+1, y=Y+1$ (3)

Again setting

$$X=M+N, Y=M-N \tag{4}$$

in (2) it simplifies to the equation

$$M^2=7N^2+4 \tag{5}$$

Now, consider the Pellian equation

$$M^2=7N^2+1 \tag{6}$$

whose general solution $\left(\tilde{N}_n, \tilde{M}_n\right)$ is given by

$$\tilde{N}_n = \frac{1}{2\sqrt{7}}\left[\left(8+3\sqrt{7}\right)^{n+1} - \left(8-3\sqrt{7}\right)^{n+1}\right],$$

$$\tilde{M}_n = \frac{1}{2}\left[\left(8+3\sqrt{7}\right)^{n+1} + \left(8-3\sqrt{7}\right)^{n+1}\right], \qquad \text{n=0, 1, 2…}$$

Thus, the general solution $\left(N_n, M_n\right)$ of (5) is given by

$$N_n = 2\tilde{N}_n = \frac{1}{\sqrt{7}}\left[\left(8+3\sqrt{7}\right)^{n+1} - \left(8-3\sqrt{7}\right)^{n+1}\right]$$

$$M_n = 2\tilde{M}_n = \left[\left(8+3\sqrt{7}\right)^{n+1} + \left(8-3\sqrt{7}\right)^{n+1}\right]$$

Taking advantage of (3) and (4), the sequence of integral solutions of (1) can be written as

$$x_n = M_n + N_n + 1 = 2\tilde{M}_n + 2\tilde{N}_n + 1 \tag{7}$$

$$y_n = M_n - N_n + 1 = 2M_n - 2\tilde{N}_n + 1 \;,\; n = 0,1,2.... .... \tag{8}$$

Thus (7) and (8) represent the non-zero distinct integral solutions of (1).

The above values of x_n and y_n satisfy respectively the following recurrence relations.

$$x_{n+2} - 16x_{n+2} + x_n = -14, \tag{9}$$

$$y_{n+2} - 16y_{n+1} + y_n = -14, \quad n = 0,1,2.... \tag{10}$$

A few numerical examples are given below

n	x_n	y_n
0	23	11
1	351	159
2	5579	2519
3	88899	40131
4	1416791	639563

Some relations satisfied by the solutions (7) and (8) are as follows:

1. Both the values of x, y are positive and odd.
2. $18x_n - 8y_n - 2y_{n+1} \equiv 0 \pmod 8$
3. $20x_n - 9y_n - x_{n+1} \equiv T_{10,2}$
4. $2\left(x_{3n+2} + y_{3n+2} + 3x_n + 3y_n - 8\right) = (x_n + y_n - 2)\left(29y_{2n+2} - 13x_{2n+2} - 12\right)$
5. $2[29y_{3n+3} - 13x_{3n+3} + 87y_{n+1} - 39x_{n+1} - 64] - (x_n + y_n - 2)\left(29y_{2n+2} - 13x_{2n+2} - 12\right) = 0$
6. $(x_n + y_n - 2)(29y_{3n+3} - 13x_{3n+3} + 87y_{n+1} - 39x_{n+1} - 64) - 2(x_{4n+3} + y_{4n+3} + 4x_{2n+1} + 4y_{2n+1} + 2) = 0$
7. $\left(x_{2n+1} + y_{2n+1} + 2\right)\left(29y_{2n+2} - 13x_{2n+2} - 12\right) = 2(29y_{4n+4} - 13x_{4n+4} + 116y_{2n+2} - 52x_{2n+2} - 68)$
8. $28\left(x_{2n+1} - y_{2n+1}\right)^2 - \left(x_{2n+1} + y_{2n+1} - 6\right)(x_n + y_n - 2) = 0$
9. Each of the following is a nasty number:

   (a) $3\left(x_{2n+1} + y_{2n+1} + 2\right)$

(b) $3(29y_{2n+2} - 13x_{2n+2} - 12)$

(c) $3(x_{2n+1} + y_{2n+1} - 6)$

(d) $3(29y_{2n+2} - 13x_{2n+2} - 20)$

10. Each of the following is a Cubical integer:

(a) $4(x_{3n+2} + y_{3n+2} + 3x_n + 3y_n - 8)$

(b) $4(29y_{3n+3} - 13y_{3n+3} + 87y_{n+1} - 39x_{n+1} - 64)$

11. Each of the following is a biquadratic integer:

(a) $8[x_{4n+3} + y_{4n+3} + 4x_{2n+1} + 4y_{2n+1} + 2]$

(b) $8[29y_{4n+4} - 13x_{4n+4} + 116y_{2n+2} - 52x_{2n+2} - 68]$

Remarkable observations

I. By considering suitable linear transformations between the solutions of (1), one may get integer solutions for the Parabola.

(a) Illustration1: It is to be noted that the Parabola

$$Y^2 = 2X$$

is satisfied for the following three sets of values of X and Y

Set1:

$$Y = 29y_{n+1} - 13x_{n+1} - 16$$
$$X = x_{2n+1} + y_{2n+1} + 2$$

Set2:

$$Y = 29y_{2n+2} - 13x_{2n+2} - 12$$
$$X = 29y_{4n+4} - 13x_{4n+4} + 116y_{2n+2} - 52x_{2n+2} - 68$$

Set3:

$$Y = x_{2n+1} + y_{2n+1} + 2$$
$$X = x_{4n+3} + y_{4n+3} + 4x_{2n+1} + 4y_{2n+1} + 2$$

(b) Illustration2: The Parabola

$$7Y^2 = 2X$$

is satisfied for the following set of values of X and Y

$$Y = 5x_{n+1} - 11y_{n+1} + 6$$
$$X = x_{2n+1} + y_{2n+1} - 6$$

II. If (x_0, y_0) is any given solution of (1), then each of the following expressions satisfies (1):

$(-y_0 + 2, -x_0 + 2), (-9x_0 + 4y_0 + 6, -20x_0 + 9y_0 + 12)$,

CONCLUSION

In conclusion one may search for other patterns of solutions and their corresponding properties.

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