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Homogeneous Bi-Quadratic Equation with Four Unknowns $(x+y)(x^3+y^3)=52z^2w^2$

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Abstract: We obtain infinitely many non-zero integer quadruples satisfying the the Biquadratic equation with four unknowns. Various interesting properties among the values of x, y, z and w are presented.
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INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, biquadratic diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-5]. In this context, one may refer [6-11] for various problems on the biquadratic diophantine equations with four variables. However, often we come across non-homogeneous biquadratic equations and as such one may require its integral solution in its most general form. It is towards this end, this paper concerns with the problem of determining non-trivial integral solutions of the homogeneous equation with four unknowns given by $(x+y)(x^3+y^3)=52z^2w^2$. A few relations among the solutions are presented.

NOTATIONS

$$t_{(m,n)}=n\left(1+\frac{(n-1)(m-2)}{2}\right).$$

$$P_n^m=\left(\frac{n(n+1)}{6}\right)[(m-2)n+(5-m)]$$

$$(OH)_n=\frac{1}{3}n(2n^3+1)$$

$$\Pr_n=n(n+1)$$

METHOD OF ANALYSIS

The homogeneous bi-quadratic equation with four unknowns to be solved is

$$(x+y)(x^3+y^3)=52z^2w^2 \tag{1}$$

Introducing the linear transformations

$$x=u+v,\ y=u-v,\ z=2u \tag{2}$$

in (1) , it is written as

$$u^2+3v^2=52w^2 \tag{3}$$

Equation (3) is solved through five different ways and thus, in view of (2), we obtain five different patterns of non-zero distinct integer solutions to (1).

Pattern 1

Let

$$w = a^2 + 3b^2 \tag{4}$$

Write 52 as

$$52 = \left(7 + i\sqrt{3}\right)\left(7 - i\sqrt{3}\right) \tag{5}$$

Using (4) & (5) in (3) and employing the method of factorization , define

$$u + i\sqrt{3}v = \left(7 + i\sqrt{3}\right)\left(a + i\sqrt{3}b\right)^2$$

Equating real and imaginary parts in the above equation , we get

$$u = 7a^2 - 21b^2 - 6ab$$
$$v = a^2 - 3b^2 + 14ab$$

Substituting the values of u , v in (2) ,we have

$$\begin{aligned} x &= 8a^2 - 24b^2 + 8ab \\ y &= 6a^2 - 18b^2 - 20ab \\ z &= 14a^2 - 42b^2 - 12ab \end{aligned} \tag{6}$$

Thus (4) & (6) represent the non-zero distinct integer solutions of (1).

Properties

- $x(a,1) - t_{(18,a)} \equiv -9 \pmod{15}$
- $y(a,1) - t_{(14,a)} \equiv -3 \pmod{15}$
- $y(a,2a^2+1) + 72t_{(4,a^2)} + 60(OH)_a + t_{(134,a)} \equiv -18 \pmod{65}$
- $x(a+1,a^2) + 24t_{(4,a^2)} - 16p_a^5 - t_{(18,a)} \equiv 8 \pmod{23}$
- $z(a,1) - t_{(28,a)} + 42$ is a perfect square.

PATTERN 2

Instead of (5) , we write 52 as

$$52 = \left(2 + i4\sqrt{3}\right)\left(2 - i4\sqrt{3}\right) \tag{7}$$

Using (4) & (7) in (3) and employing the method of factorization define

$$\left(u + i\sqrt{3}v\right) = \left(2 + i\sqrt{3}\right)\left(a + i\sqrt{3}b\right)^2$$

Equating the real and imaginary parts in the above equation , we get

$$u = 2a^2 - 6b^2 - 24ab$$
$$v = 4a^2 - 12b^2 + 4ab$$

Substituting the values of u, v in (2) ,we have

$$\begin{aligned} x &= 6a^2 - 18b^2 - 20ab \\ y &= -2a^2 + 6b^2 - 28ab \\ z &= 4a^2 - 12b^2 - 48ab \end{aligned} \tag{8}$$

Thus (4) & (8) represent the non-zero distinct integer solutions of (1).

Properties

- $y(a,1)+t_{(6,a)}\equiv 6(\bmod 29)$
- $z(a,1)-t_{(10,a)}\equiv -12(\bmod 45)$
- $x(a+1,a^2)+18t_{(4,a^2)}+40p_a^5-t_{(14,a)}\equiv 6(\bmod 17)$
- $y(a+1,a^2)-6t_{(4,a^2)}+56p_a^5+t_{(6,a)}\equiv -2(\bmod 5)$
- $x(a,2a^2+1)+72t_{(4,a^2)}+60(OH)_a+t_{(134,a)}\equiv -18(\bmod 65)$

Pattern 3

Write (3) as

$$3v^2=52w^2-u^2 \tag{9}$$

Assume

$$v=52a^2-b^2 \tag{10}$$

Write 3 as

$$3=\left(\sqrt{52}+7\right)\left(\sqrt{52}-7\right) \tag{11}$$

Using (10) & (11) in (9)and employing the method of factorization, define

$$\sqrt{52}w+u=\left(\sqrt{52}+7\right)\left(\sqrt{52}a+b\right)^2$$

Equating the rational and irrational parts in the above equation, we get

$$\begin{aligned} w&=52a^2+b^2+14ab \\ u&=364a^2+7b^2+104ab \end{aligned} \tag{11a}$$

Substituting the values of u, w in (2) ,we have

$$\begin{aligned} x&=416a^2+6b^2+104ab \\ y&=312a^2+8b^2+104ab \\ z&=728a^2+14b^2+208ab \end{aligned} \tag{12}$$

Thus (11a) & (12) represent the non-zero distinct integer solutions of (1).

PROPERTIES:

- $x(a,1)-t_{(834,a)}\equiv 6(\bmod 519)$
- $y(a,1)-t_{(626,a)}\equiv 8(\bmod 415)$
- $x(a+1,a)-t_{(846,a)}-104p_{r_a}\equiv 416(\bmod 1253)$
- $x(a+1,a^2)-t_{(834,a)}-6t_{4,a^2}-208P_a^5\equiv 416(\bmod 1247)$
- $x(a,2a^2+1)-24t_{(4,a^2)}+312(OH)_a-t_{(882,a)}\equiv 6(\bmod 439)$
- $y(a+1,a)-104P_{ra}-t_{(642,a)}\equiv 312(\bmod 943)$

Pattern 4

Instead of (11) , we write 3 as

$$3=\frac{(\sqrt{52}+5)(\sqrt{52}-5)}{3^2} \tag{13}$$

Using (11) & (13) in (9)and employing the method of factorization ,define

$$\sqrt{52}w+u=\frac{(\sqrt{52}+5)(\sqrt{52}a+b)^2}{3}$$

Equating the rational and irrational parts in the above equation ,we get

$$u=\frac{1}{3}[260a^2+5b^2+104ab]$$
$$v=52a^2-b^2$$
$$w=\frac{1}{3}[52a^2+b^2+10ab]$$

Substituting the values of u, v in (2) & replacing a by 3a, b by 3b the corresponding non-zero distinct integer solutions to (1) are given by

$$\begin{aligned} x&=1248a^2+6b^2+312ab \\ y&=312a^2+24b^2+312ab \\ z&=1560a^2+30b^2+624ab \\ w&=156a^2+3b^2+30ab \end{aligned} \tag{14}$$

PROPERTIES:

- $y(a+1,a)-t_{(674,a)}-312p_{r_a}\equiv 312(\mathrm{mod}\,959)$
- $z(a+1,a^2)-t_{(3122,a)}-1248p_a^5-30t_{(4,a^2)}\equiv 1560(\mathrm{mod}\,4679)$
- $x(a,2a^2+1)-24t_{(4,a^2)}-936(OH)a-t_{(2558,a)}\equiv 6(\mathrm{mod}\,1277)$
- $y(a,2a^2+1)-96t_{(4,a^2)}-936(OH)_a-t_{(818,a)}\equiv 24(\mathrm{mod}\,407)$
- $x(a+1,a)-312p_{r_a}-t_{(998,a)}-t_{(518,a)}\equiv 1248(\mathrm{mod}\,3748)$

Pattern 5

In addition to (13) & (11) , we write 3 as

$$3=\frac{(\sqrt{52}+2)(\sqrt{52}-2)}{4^2} \tag{15}$$

Using (10) & (15) in (9)and employing the method of factorization , define

$$\sqrt{52}w+u=\frac{(\sqrt{52}+2)(\sqrt{52}a+b)^2}{4}$$

Equating the rational and irrational parts in the above equation ,we get

$$u=\frac{1}{4}[104a^2+2b^2+104ab]$$
$$w=\frac{1}{4}[52a^2+b^2+4ab]$$

Substituting the values of u, v in (2) & replacing a by 4a, b by 4b the corresponding non-zero distinct integer solutions to (1) are given by

$$\begin{aligned} x &= 1248a^2 - 8b^2 + 416ab \\ y &= -416a^2 + 24b^2 + 416ab \\ z &= 832a^2 + 16b^2 + 832ab \\ w &= 208a^2 + 4b^2 + 16ab \end{aligned} \tag{16}$$

Properties

- $x(a,2a^2+1) + 32t_{(4,a^2)} + 1248(OH)a - t_{(2434a)} \equiv -8(\mathrm{mod}\,1215)$
- $y(a,2a^2+1) - 96t_{(4,a^2)} - 1248(OH)a + t_{(642,a)} \equiv 24(\mathrm{mod}\,319)$
- $z(a,2a^2+1) - 64t_{(4,a^2)} - 2496(OH)a - t_{(1794a)} \equiv 16(\mathrm{mod}\,895)$
- $y((a+2),a(a+1)) - 2496p_a^3 - 24t_{(4,a^2)} - 96p_a^5 + t_{(882,a)} \equiv -1664(\mathrm{mod}\,2103)$

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