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The exact single traveling wave solutions to the Gardner equation

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Abstract: Using the complete discrimination system for polynomial, we give the classification of single traveling wave solutions to the Gardner equation.
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INTRODUCTION

In the paper, we consider the Gardner equation[1,2],

$$u_t = (\beta w - \frac{\beta^2}{2}u^2 + \delta u)u_x + w_y + \varepsilon^2 u_{xxx}, \tag{1}$$

where $w_x = v_y$. We will give the classification of single traveling wave solutions to the Gardner equation by complete discrimination system for polynomial method[3,4].

EXACT SOLUTIONS

In order to obtain the exact traveling wave solutions, we take a wave transformation $u = u(\xi)$ and $\xi = k_1 x + k_2 y + ct$. As $w_x = v_y$, we have $w = \frac{k_2}{k_1}u + G_1$,(G_1 is an integral constant). The Gardner equation is reduced to the following ODE,

$$cu' = \beta k_2 uu' + \beta G_1 k_1 u' - \frac{\beta^2}{2}k_1 u^2 u' + k_1 \delta uu' + \frac{k_2^2}{k_1}u' + \varepsilon^2 k_1^3 u'''. \tag{2}$$

Multiplying the both sides of the Eq.(3) by u' and integrating it once, we can have:

$$(u')^2 = a_4 u^4 + a_3 u^3 + a_2 u^2 + a_0, \tag{3}$$

where $a_4 = \frac{\beta^2}{6\varepsilon^2 k_1^2}$, $a_3 = -\frac{2(\beta k_2 + \delta k_1)}{3\varepsilon^2 k_1^3}$, $a_2 = \frac{2(k_1 c - \beta G_1 k_1^2 - k_2^2)}{\varepsilon^2 k_1^4}$, $a_0 = G_2$, G_2 is an integral constant. The solutions of u can be given from

$$\pm(\xi - \xi_0) = \int \frac{du}{\sqrt{a_4 u^4 + a_3 u^3 + a_2 u^2 + a_0}}. \tag{4}$$

In order to solve the Eq.(4), we take the transformation as $y = (a_4)^{\frac{1}{4}}(u + \frac{a_3}{4a_4})$ when $a_4 > 0$, then the Eq.(4) yields

$$\pm(a_4)^{\frac{1}{4}}(\xi - \xi_0) = \frac{dy}{\sqrt{y^4 + py^2 + qy + r}}, \tag{5}$$

where $p = \frac{a_2}{\sqrt{a_4}}$, $q = (\frac{a_3^3}{8a_4^2} - \frac{a_2 a_3}{2a_4})(a_4)^{-\frac{1}{4}}$, $r = \frac{-3a_3^4}{256a_4^3} + \frac{a_2 a_3^2}{16a_4^2} + a_0$. If $a_4 < 0$, then we take the transformation as $y = (-a_4)^{\frac{1}{4}}(u + \frac{a_3}{4a_4})$, the Eq.(4) becomes

$$\pm(a_4)^{\frac{1}{4}}(\xi-\xi_0)=\frac{dy}{\sqrt{-(y^4+py^2+qy+r)}}, \tag{6}$$

where $p=\frac{-a_2}{\sqrt{-a_4}}$, $q=(-\frac{a_3^3}{8a_4^2}+\frac{a_2a_3}{2a_4})(-a_4)^{-\frac{1}{4}}$, $r=\frac{3a_3^4}{256a_4^3}-\frac{a_2a_3^2}{16a_4^2}-a_0$. To get the solutions to the Eq.(5) and (6), we denote $F(y)=y^4+py^2+qy+r$. Its complete discrimination system [5,6] is computed as follows:

$$D_1=4, D_2=-p, D_3=-2p^3+8pr-9q^2, \tag{7}$$

$$D_4=-p^3q^2+4p^4r+36pq^2r-32p^2r^2-\frac{27}{4}q^4+64r^3, \tag{8}$$

$$E_2=9p^2-32pr. \tag{9}$$

According to the complete discrimination system for polynomial F(w), the classification of the traveling wave solutions of the Gardner equation can be discussed:

Case 1. $D_2=0, D_3=0$ and $D_4=0$. Then we have $F(y)=y^4$, here $a_4>0$. By Eq.(5), we can give the solutions

$$y=-(a_4)^{-\frac{1}{4}}(\xi-\xi_0)^{-1}. \tag{10}$$

Case 2. $D_2<0$, $D_3=0$, and $D_4=0$. $F(y)=((y-l)^2+s^2)^2$, where l , s are real numbers, and $s>0$. For $a_4>0$, we have

$$y=s\tan((a_4)^{-\frac{1}{4}}(\xi-\xi_0)s)+l. \tag{11}$$

Case 3. $D_2>0$, $D_3=0$, $D_4=0$ and $E_2>0$. Then we have $F(y)=(y-\alpha)^2(y-\beta)^2$, where α,β are real numbers, and $\alpha\neq\beta$. For $a_4>0$, we have

$$\pm(a_4)^{\frac{1}{4}}(\xi-\xi_0)=\frac{1}{\alpha-\beta}ln\left|\frac{y-\alpha}{y-\beta}\right|. \tag{12}$$

For $y>\alpha$ or $y<\beta$, by the Eq.(10)we have

$$y=\beta+\frac{\beta-\alpha}{\exp[(a_4)^{-\frac{1}{4}}(\alpha-\beta)(\xi-\xi_0)]-1}, \tag{13}$$

when $\beta<y<\alpha$, by the Eq.(10)we have

$$y=\beta-\frac{\beta-\alpha}{\exp[(a_4)^{-\frac{1}{4}}(\alpha-\beta)(\xi-\xi_0)]+1}. \tag{14}$$

Case 4. $D_2>0$, $D_3=0$, $D_4=0$ and $E_2=0$. Then we have $F(y)=(y-\alpha)^3(y-\beta)$, where α,β are real numbers, and $\alpha\neq\beta$. When $a_4>0$, we have

$$\pm(a_4)^{\frac{1}{4}}(\xi-\xi_0)=\frac{2}{\beta-\alpha}\sqrt{\frac{y-\beta}{y-\alpha}}. \tag{15}$$

When $y>\alpha$, $y>\beta$ or $y<\alpha$, $y<\beta$, the solution is

$$y=\alpha+\frac{4(\alpha-\beta)}{(a_4)^{\frac{1}{4}}(\alpha-\beta)^2(\xi-\xi_0)^2-4}. \tag{16}$$

For $a_4<0$, we have

$$\pm(a_4)^{\frac{1}{4}}(\xi-\xi_0)=\frac{2}{\alpha-\beta}\sqrt{\frac{\beta-y}{y-\alpha}}. \tag{17}$$

When $y>\alpha$, $y<\beta$ or $y<\alpha$, $y>\beta$, we can get a solitary solution as

$$y=\alpha-\frac{4(\alpha-\beta)}{(a_4)^{\frac{1}{4}}(\alpha-\beta)^2(\xi-\xi_0)^2+4}. \tag{18}$$

Case 5. $D_2D_3<0$, and $D_4=0$. $F(y)=(y-\alpha)((y-l)^2+s^2)$. By Eq.(5), we have

$$y=\frac{[e^{\pm(a_4)^{\frac{1}{4}}m(\xi-\xi_0)}-\frac{\alpha-2l}{m}]+[2m-\alpha+2l]}{[e^{\pm(a_4)^{\frac{1}{4}}m(\xi-\xi_0)}-\frac{\alpha-2l}{m}]^2-1}, \tag{19}$$

where $m=\sqrt{(\alpha-l)^2+s^2}$.

Case 6. For $D_2>0$, $D_3>0$, $D_4>0$ and other cases, the corresponding solutions can be expressed by hyper-elliptic functions or hyper-elliptic integral. We omit them for simplicity.

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