Integer Points on the Hyperbola \( x^2 - 4xy + y^2 + 16x = 0 \)

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Abstract: The binary quadratic equation \( x^2 - 4xy + y^2 + 16x = 0 \) representing hyperbola is considered. Different patterns of solutions are obtained. A few relations between the solutions are exhibited.

Keywords: Binary quadratic, Hyperbola, Integer solutions.

INTRODUCTION
There is an unlimited field of research in binary quadratic equations because of their large variety [1-5]. There are some already available literature in the field of binary quadratic equations [6-19]. This communication concerns with yet another interesting binary quadratic equation \( x^2 - 4xy + y^2 + 16x = 0 \) for determining its infinitely many non-zero integral solutions. Also a few interesting relations between the solutions are presented.

NOTATIONS
Polygonal Number of rank n with size m
\[ t_{m,n} = n \left( 1 + \frac{(n-1)(m-2)}{2} \right) \]

Pentagonal pyramidal number of rank n
\[ P_n^5 = \frac{n^2(n+1)}{2} \]

Pronic number of rank n
\[ Pr_n = n(n + 1) \]

METHOD OF ANALYSIS
The hyperbola under consideration is
\[ x^2 - 4xy + y^2 + 16x = 0 \]  (1)

To start with, it is seen that (1) is satisfied by the following pairs of integers \( (8,8),(8,24),(-16,-64),(72,24),(-256,-64) \).

However, we have other choices of solutions satisfying (1) and they are illustrated below:
Treating (1) as a quadratic in x and solving for x, we get
\[ x = (2y - 8) \pm \sqrt{3y^2 - 32y + 64} \]  (2)

Let \( \alpha^2 = 3y^2 - 32y + 64 \) \nand substituting \( y = \frac{Y + 16}{3} \) \n(3)
in (3), we have \( Y^2 = 3\alpha^2 + 8^2 \) \n(4)
Consider the Pellian equation
\[ Y^2 = 3\alpha^2 + 1 \]
whose general solution is given by
\[ Y_n = \frac{1}{2} \left[ \left( 2 + \sqrt{3} \right)^{n+1} + \left( 2 - \sqrt{3} \right)^{n+1} \right] \]  (5)

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From (4) and (5), we have the general solutions of the equation
\[ y_n = \frac{4}{3} \left[ \left(2 + \sqrt{3}\right)^{n+1} + \left(2 - \sqrt{3}\right)^{n+1} \right] + \frac{16}{3} \]  
(8)

Substituting (7) and (8) in (2) and taking the negative sign, the corresponding integer solutions to (1) are given by
\[ x_n = \frac{4}{3} \left[ \left(2 + \sqrt{3}\right)^{n} + \left(2 - \sqrt{3}\right)^{n} \right] + \frac{8}{3}, \quad n=1,3,5,.... \]
\[ y_n = \frac{4}{3} \left[ \left(2 + \sqrt{3}\right)^{n+1} + \left(2 - \sqrt{3}\right)^{n+1} \right] + \frac{16}{3}, \quad n=1,3,5,.... \]

Some numerical examples are presented below:

<table>
<thead>
<tr>
<th>n</th>
<th>x_n</th>
<th>y_n</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>24</td>
</tr>
<tr>
<td>3</td>
<td>72</td>
<td>264</td>
</tr>
<tr>
<td>5</td>
<td>968</td>
<td>3608</td>
</tr>
<tr>
<td>7</td>
<td>13448</td>
<td>50184</td>
</tr>
</tbody>
</table>

Also, taking the positive sign in (2), the other set of solutions to (1) is given by
\[ x_n = \frac{4}{3} \left[ \left(2 + \sqrt{3}\right)^{n+2} + \left(2 - \sqrt{3}\right)^{n+2} \right] + \frac{8}{3}, \quad n=1,3,5,... \]
\[ y_n = \frac{4}{3} \left[ \left(2 + \sqrt{3}\right)^{n+1} + \left(2 - \sqrt{3}\right)^{n+1} \right] + \frac{16}{3}, \quad n=1,3,5,... \]

Properties

- \(3x_{2n}\) is a square integer
- \(x_{n+4} - 14x_{n+2} + x_n = -32\)
- \(y_{n+4} - 14y_{n+2} + y_n = -64\)

Alternatively, treating (1) as a quadratic in \(y\) and solving for \(y\), we get
\[ y = 2x \pm \sqrt{3x^2 - 16x} \]  
(9)

Let \(\alpha^2 = 3x^2 - 16x\)  
(10)

And substituting \(x = \frac{X + 8}{3}\)  
(11)

in (9), we have
\[ X^2 = 3\alpha^2 + 8^2 \]  
(12)

whose general solution of the pellian equation
\[ X^2 = 3\alpha^2 + 1 \]

is given by
\[ X_n = \frac{1}{2} \left[ \left(2 + \sqrt{3}\right)^{n+1} + \left(2 - \sqrt{3}\right)^{n+1} \right] \]  
(13)

\[ \alpha_n = \frac{1}{2\sqrt{3}} \left[ \left(2 + \sqrt{3}\right)^{n+1} - \left(2 - \sqrt{3}\right)^{n+1} \right] \]  
(14)

From (10) and (12) we have
\[ x_n = \frac{4}{3} \left[ \left(2 + \sqrt{3}\right)^{n+2} + \left(2 - \sqrt{3}\right)^{n+2} \right] + \frac{8}{3} \]  
(15)

Substituting (13) and (14) in (9) and taking the positive sign, the corresponding integer solutions to (1) are given by
\[ x_n = \frac{4}{3} \left[ \left(2 + \sqrt{3}\right)^{n+1} + \left(2 - \sqrt{3}\right)^{n+1} \right] + \frac{8}{3}, \quad n=0,2,4,... \]
\[ y_n = \frac{4}{3} \left[ \left(2 + \sqrt{3}\right)^{n+2} + \left(2 - \sqrt{3}\right)^{n+2} \right] + \frac{16}{3}, \quad n=0,2,4,.. \]
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</tbody>
</table>

Also, taking the negative sign in (9), the other set of solutions to (1) is given by

\[
x_n = \frac{4}{3} \left[ (2 + \sqrt{3})^{n+1} + (2 - \sqrt{3})^{n+1} \right] + \frac{8}{3} \quad n = 0, 2, 4, \ldots
\]

\[
y_n = \frac{4}{3} \left[ (2 + \sqrt{3})^{n} + (2 - \sqrt{3})^{n} \right] + \frac{16}{3} \quad n = 0, 2, 4, \ldots
\]

Properties

\( x_{n+4} - 14x_{n+2} + x_n = -32 \)

\( y_{n+4} - 14y_{n+2} + y_n = -64 \)

CONCLUSION

As the binary quadratic equations representing hyperbolas are rich in variety, one may consider other forms of hyperbolas and search for their non-trivial distinct integral solutions along with the corresponding properties.

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