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# Integer Points on the Hyperbola $x^2-4xy+y^2+16x=0$

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**Abstract:** The binary quadratic equation $x^2-4xy+y^2+16x=0$ representing hyperbola is considered. Different patterns of solutions are obtained. A few relations between the solutions are exhibited.
**Keywords:** Binary quadratic, Hyperbola, Integer solutions.

## INTRODUCTION

There is an unlimited field of research in binary quadratic equations because of their large variety [1-5]. There are some already available literature in the field of binary quadratic equations [6-19] .This communication concerns with yet another interesting binary quadratic equation $x^2-4xy+y^2+16x=0$ for determining its infinitely many non-zero integral solutions. Also a few interesting relations between the solutions are presented.

## NOTATIONS

Polygonal Number of rank n with size m

$$t_{m,n}=n\left(1+\frac{(n-1)(m-2)}{2}\right)$$

Pentagonal pyramidal number of rank n

$$P_n^5=\frac{n^2(n+1)}{2}$$

Pronic number of rank n

$$Pr_n=n(n+1)$$

## METHOD OF ANALYSIS

The hyperbola under consideration is

$$x^2-4xy+y^2+16x=0 \tag{1}$$

To start with, it is seen that (1) is satisfied by the following pairs of integers (8,8),(8,24),(-16,-64),(72,24),(-256,-64).

However, we have other choices of solutions satisfying (1) and they are illustrated below:
Treating (1) as a quadratic in x and solving for x, we get

$$x=(2y-8)\pm\sqrt{3y^2-32y+64} \tag{2}$$

Let $\alpha^2=3y^2-32y+64$ (3)

and substituting $y=\frac{Y+16}{3}$ (4)

in (3), we have $Y^2=3\alpha^2+8^2$ (5)

Consider the Pellian equation

$$Y^2=3\alpha^2+1$$

whose general solution is given by

$$\tilde{Y}_n=\frac{1}{2}\left[\left(2+\sqrt{3}\right)^{n+1}+\left(2-\sqrt{3}\right)^{n+1}\right] \tag{6}$$

$$\tilde{\alpha}_n = \frac{1}{2\sqrt{3}}\left[\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right] \quad (7)$$

From (4) and (5) , we have the general solutions of the equation

$$y_n = \frac{4}{3}\left[\left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}\right] + \frac{16}{3} \quad (8)$$

Substituting (7) and (8) in (2) and taking the negative sign, the corresponding integer solutions to (1) are given by

$$x_n = \frac{4}{3}\left[\left(2+\sqrt{3}\right)^{n} + \left(2-\sqrt{3}\right)^{n}\right] + \frac{8}{3}, \quad n=1,3,5,\ldots..$$

$$y_n = \frac{4}{3}\left[\left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}\right] + \frac{16}{3}, \quad n=1,3,5,\ldots.$$

Some numerical examples are presented below:

| n | $x_n$ | $y_n$ |
|---|---|---|
| 1 | 8 | 24 |
| 3 | 72 | 264 |
| 5 | 968 | 3608 |
| 7 | 13448 | 50184 |

Also, taking the positive sign in (2), the other set of solutions to (1) is given by

$$x_n = \frac{4}{3}\left[\left(2+\sqrt{3}\right)^{n+2} + \left(2-\sqrt{3}\right)^{n+2}\right] + \frac{8}{3}, n = 1,3,5\ldots$$

$$y_n = \frac{4}{3}\left[\left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}\right] + \frac{16}{3} n = 1,3,5\ldots$$

**Properties**

- ❖ $3x_{2n}$ is a square integer
- ❖ $x_{n+4} - 14x_{n+2} + x_n = -32$
- ❖ $y_{n+4} - 14y_{n+2} + y_n = -64$

Alternatively, treating (1) as a quadratic in y and solving for y, we get

$$y = 2x \pm \sqrt{3x^2 - 16x} \quad (9)$$

Let $\alpha^2 = 3x^2 - 16x$ (10)

And substituting $x = \frac{X+8}{3}$ (11)

in (9), we have $X^2 = 3\alpha^2 + 8^2$ (12)

whose general solution of the pellian equation

$$X^2 = 3\alpha^2 + 1$$

is given by

$$\tilde{X}_n = \frac{1}{2}\left[\left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}\right] \quad (13)$$

$$\tilde{\alpha}_n = \frac{1}{2\sqrt{3}}\left[\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right] \quad (14)$$

From (10) and (12) we have

$$x_n = \frac{4}{3}\left[\left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}\right] + \frac{8}{3} \quad (15)$$

Substituting (13) and (14) in (9) and taking the positive sign, the corresponding integer solutions to (1) are given by

$$x_n = \frac{4}{3}\left[\left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}\right] + \frac{8}{3}, n = 0,2,4\ldots.$$

$$y_n = \frac{4}{3}\left[\left(2+\sqrt{3}\right)^{n+2} + \left(2-\sqrt{3}\right)^{n+2}\right] + \frac{16}{3}, \quad n = 0,2,4..$$

Some numerical examples are presented below:

| n | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 8 | 24 |
| 2 | 72 | 264 |
| 4 | 968 | 3608 |
| 6 | 13448 | 50184 |

Also, taking the negative sign in (9), the other set of solutions to (1) is given by

$$x_n = \frac{4}{3}\left[\left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}\right] + \frac{8}{3}, n = 0,2,4....$$

$$y_n = \frac{4}{3}\left[\left(2+\sqrt{3}\right)^{n} + \left(2-\sqrt{3}\right)^{n}\right] + \frac{16}{3}, \; n = 0,2,4..$$

**Properties**

- $x_{n+4} - 14x_{n+2} + x_n = -32$
- $y_{n+4} - 14y_{n+2} + y_n = -64$

**CONCLUSION**

As the binary quadratic equations representing hyperbolas are rich in variety, one may consider other forms of hyperbolas and search for their non-trivial distinct integral solutions along with the corresponding properties.

**REFERENCES**


1. Dickson LE; History of Theory of Numbers. Volume 2, Chelsea Publishing company, New York, 1952.
2. Mordell LJ; Diophantine Equations. Academic Press, London, 1969.
3. Smart NP; the algorithmic Resolutions of Diophantine equations, Cambridge University Press, 1999.
4. Smith DE; History of mathematics. Volume I and II, Dover Publications, New York, 1953.
5. Gopalan MA, Vidhyalakshmi S, Devibala S; On the Diophantine equation $3x^2 + xy = 14$. Acta Ciencia Indica, 2007; XXXIIIM(2): 645-646.
6. Gopalan MA, Janaki G; Observations on $Y^2 = 3X^2 + 1$. Acta Ciencia Indica, 2008; XXXIVM(2): 693.
7. Gopalan MA, Vijayalakshmi R; Special Pythagorean triangles generated through the integral solutions of the equation $y^2 = (K^2 + 1)x^2 + 1$. Antarctica J.Math., 2010; 7(5): 503-507.
8. Gopalan MA, Sivagami B; Observations on the integral solutions of $y^2 = 7x^2 + 1$. Antartica J Math., 2010; 7(3): 291-296.
9. Gopalan MA, Vijayalakshmi R; Observation on the integral solutions of $y^2 = 5x^2 + 1$. Impact J Sci Tech., 4(4): 125-129.
10. Gopalan MA, Sangeetha G; A remarkable observation on $y^2 = 10x^2 + 1$. Impact J Sci Tech., 2010; 4(1): 103-106.
11. Gopalan MA, Parvathy G; Integral points on the Hyperbola $x^2 + 4xy + y^2 - 2x - 10y + 24 = 0$. Antarctica J.Math., 2010; 7(2): 149-155.
12. Gopalan MA, Palanikumar R; Observations on $y^2 = 12x^2 + 1$. Antarctica J Math., 2011; 8(2): 149-152.
13. Gopalan MA, Devibala S, Vijayalakshmi R; Integral points on the hyperbola $2x^2 - 3y^2 = 5$. American Journal of Applied Mathematics and Mathematical Sciences, 2012; I(1): 1-4.
14. Gopalan MA, Vidhyalakshmi S, Usha Rani TR, Mallika S; Observations on $y^2 = 12x^2 - 3$. Bessel J Math., 2012; 2(3): 153-158.
15. Gopalan MA, Vidhyalakshmi S, Sumathi G, Lakshmi K; Integral points on the Hyperbola $x^2 + 6xy + y^2 + 40x + 8y + 40 = 0$. Bessel J Math., 2012; 2(3): 159-164.
16. Gopalan MA, Geetha K; Observations on the Hyperbola $y^2 = 18x^2 + 1$. Retell., 2012; 13(1): 81-83.
17. Gopalan MA, Sangeetha G, Somanath M; Integral points on the Hyperbola $(a+2)x^2 - ay^2 = 4a(k-1) + 2k^2$. Indian Journal of Science, 2012; I(2): 125-126.
18. Gopalan MA, Vidhyalakshmi S, Kavitha A; Observations on the Hyperbola $ax^2 - (a+1)y^2 = 3a - 1$. Discovery, 2013; 4(10): 22-24.
19. Gopalan MA, MeenaK, Vidhyalakshmi S, Thangam SA, Premalatha E; Integer points on the hyperbola $x^2 - 6xy + y^2 + 4x = 0$. Sch J Eng Tech., 2014; 2(1): 14-18.