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On Strong Rational Diophantine Quadruples with Equal Members

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Abstract: This paper concerns with the study of constructing strong rational Diophantine quadruples from two given equal members.

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INTRODUCTION

Let q be a non-zero rational number. A set $\{a_1, a_1, \dots, a_m\}$ of non-zero rational is called a rational $d(q)$ – m-tuple, if $a_i a_j + q$ is a square of a rational number for all $1 \le i \le j \le m$.The Greek mathematician Diophantus of Alexandria considered a variety of problems on in determinant equations with rational or integer solutions. In particular, one of the problems was to find the sets of distinct positive rational numbers such that the product of any two numbers is one less than a rational square [14] and Diophantus found four positive rationales $\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}$ [4,5]. The first set of four positive integers with the same property, the set $\{1,3,8,120\}$ was found by Fermat. It was proved in 1969 by Baker and Davenport [3] that a fifth positive integer cannot be added to this set and one may refer [6, 7, 11] for generalization. However, Euler discovered that a fifth rational number can be added to give the following rational Diophantine quintuple $\left\{1,3,8,120,\frac{777480}{8288641}\right\}$.Rational sextuples with two equal elements have been given in [2]. In 1999,Gibs [13] found several examples of rational Diophantine sextuples, e.g., $\left\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\right\}$, $\left\{\frac{17}{448}, \frac{265}{448}, \frac{2145}{448}, 252, \frac{23460}{7}, \frac{2352}{7921}\right\}$. All known Diophantine quadruples are regular and it has been conjectured that there are no irregular Diophantine quadruples [1,13] (this is known to be true for polynomials with integer co-efficient [8]).If so, then there are no Diophantine quintuples. However there are infinitely many irregular rational Diophantine quadruples. The smallest is $\frac{1}{4}, 5, \frac{33}{4}, \frac{105}{4}$**.** Many of these irregular quadruples are examples of another common type for which two of the sub-triples are regular i.e., $\{a, b, c, d\}$ is an irregular rational Diophantine quadruple, while $\{a, b, c,\}$ and $\{a, b, d\}$ are regular Diophantine triples. These are known as semi-regular rational Diophantine quadruples. There are only finitely many of these for any given common denominator l and they can be readily found. Moreover in [12], it has been proved that the $D(\mp k^2)$- triple $\{k^2, k^2 \pm 1, 4k^2 \pm 1\}$ cannot be extended to a $D(\mp k^2)$- quintuple. In [10], it has been proved that $D(-k^2)$- triple $\{1, k^2+1, k^2+4\}$ cannot be extended to a $D(-k^2)$- quadruple if $k \ge 5$. These results motivated us to search for strong rational Diophantine quadruples generated from two equal members.

STRONG RATIONAL DIOPAHANTINE QUADRUPLES WITH EQUAL MEMBERS

Theorem 1

Let $A = \frac{1}{k}, B = \frac{1}{k}, C = \frac{1}{4k} - k$ and $D = \frac{4k^2+9}{4k}$ .Then (A, B, C, D) is a $d(k^2+2)$ –strong rational Diophantine quadruple.

Proof

$$AB+(k^2+2)=\frac{1}{k^2}+(k^2+2)$$

$$=\left(\frac{k^2+1}{k}\right)^2$$

$$AC+(k^2+2)=\frac{1}{k}\left(\frac{1}{4k}-k\right)+(k^2+2)$$

$$=\left(\frac{2k^2+1}{2k}\right)^2=BC+(k^2+2) \qquad \text{(since A=B)}$$

$$AD+(k^2+2)=\frac{1}{k}\left(\frac{4k^2+9}{4k}\right)+(k^2+2)$$

$$=(2k^2+3)^2=BD+(k^2+2)$$

$$CD+(k^2+2)=\left(\frac{1}{4k}-k\right)\frac{4k^2+9}{4k}+(k^2+2)$$

$$=\left(\frac{3}{4k}\right)^2 \quad \text{Hence proved}$$

Some examples are presented below

k	**(A,B,C,D)**	**Property** $d(k^2+2)$
1	$\left(1,1,\frac{-3}{4},\frac{13}{4}\right)$	$d(3)$
2	$\left(\frac{1}{2},\frac{1}{2},\frac{-15}{8},\frac{25}{8}\right)$	$d(6)$
3	$\left(\frac{1}{3},\frac{1}{3},\frac{-35}{12},\frac{45}{12}\right)$	$d(11)$
4	$\left(\frac{1}{4},\frac{1}{4},\frac{-63}{16},\frac{73}{16}\right)$	$d(18)$
5	$\left(\frac{1}{5},\frac{1}{5},\frac{-99}{20},\frac{109}{20}\right)$	$d(27)$
$(1+i2)$	$\left(\frac{1}{1+i2},\frac{1}{1+i2},\frac{13-16i}{4+8i},\frac{16i-3}{8i+4}\right)$	$d\left(\frac{8i-5}{4i-3}\right)$
$(1+i\sqrt{5})$	$\left(\frac{1}{(1+i\sqrt{5})},\frac{1}{(1+i\sqrt{5})},\frac{17-i8\sqrt{5}}{4+i4\sqrt{5}},\frac{i8\sqrt{5}-7}{4+i4\sqrt{5}}\right)$	$d(i2\sqrt{5}-2)$
$(1+2\sqrt{3})$	$\left(\frac{1}{1+2\sqrt{3}},\frac{1}{1+2\sqrt{3}},\frac{-(51+16\sqrt{3})}{(1+2\sqrt{3})},\frac{4i-3}{8i+4}\right)$	$d(15+4\sqrt{3})$

Theorem 2

Let $A=\frac{1}{k}, B=\frac{1}{k}, C=2k^2-k$ and $D=(2k^2+k+2)+\frac{1}{k}$ .Then (A,B,C,D) is a $d(k^2+2)$ –strong rational Diophantine quadruple.

Proof

$$AB+(k^2+2)=\frac{1}{k^2}+(k^2+2)$$
$$=\left(\frac{k^2+1}{k}\right)^2$$
$$AC+(k^2+2)=\frac{1}{k}\left(2k^2-k\right)+(k^2+2)$$
$$=(k+1)^2=BC+(k^2+2) \qquad \text{(since A=B)}$$
$$AD+(k^2+2)=\frac{1}{k}\left((2k^2+k+2)+\frac{1}{k}\right)+(k^2+2)$$
$$=(k^2+k+1)^2=BD+(k^2+2)$$
$$CD+(k^2+2)=(2k^2-k)[(2k^2+k+2)+\frac{1}{k}]+(k^2+2)$$
$$=(2k^2+1)^2 \text{ Hence proved}$$

Some examples are presented below

k	**(A,B,C,D)**	**Property** $d(k^2+2)$
1	$(1,1,1,6)$	$d(3)$
2	$\left(\frac{1}{2},\frac{1}{2},6,\frac{25}{2}\right)$	$d(6)$
3	$\left(\frac{1}{3},\frac{1}{3},15,\frac{70}{3}\right)$	$d(11)$
4	$\left(\frac{1}{4},\frac{1}{4},28,\frac{153}{4}\right)$	$d(18)$
5	$\left(\frac{1}{5},\frac{1}{5},45,\frac{286}{5}\right)$	$d(27)$
$(1+\sqrt{3})$	$\left(\frac{1}{1+\sqrt{3}},\frac{1}{1+\sqrt{3}},7+3\sqrt{3},\frac{27+16\sqrt{3}}{1+\sqrt{3}}\right)$	$d(6+2\sqrt{3})$
$(2+i)$	$\left(\frac{1}{(2+i)},\frac{1}{(2+i)},4+7i,\frac{12+28i}{2+i}\right)$	$d(5+4i)$
$(3+i\sqrt{2})$	$\left(\frac{1}{3+i\sqrt{2}},\frac{1}{3+i\sqrt{2}},13+i11\sqrt{2},\frac{32+i58\sqrt{2}}{3+i\sqrt{2}}\right)$	$d(9+i6\sqrt{2})$

Theorem 3

Let $A=\frac{1}{k}, B=\frac{1}{k}, C=\frac{1}{4k}+3k$ and $D=\frac{20k^2+9}{4k}$ .Then (A,B,C,D) is a $d(k^2-2)$ –strong rational Diophantine quadruple.

Proof

$$AB+(k^2-2)=\frac{1}{k^2}+(k^2-2)$$

$$=\left(\frac{k^2-1}{k}\right)^2$$

$$AC+(k^2-2)=\frac{1}{k}\left(\frac{1}{4k}+3k\right)+(k^2-2)$$

$$=\left(\frac{2k^2+1}{2k}\right)^2=BC+(k^2-2) \qquad \text{(since A=B)}$$

$$AD+(k^2-2)=\frac{1}{k}\left(\frac{20k^2+9}{4k}\right)+(k^2-2)$$

$$=\left[\frac{2k+3}{2k}\right]^2=BD+(k^2-2)$$

$$CD+(k^2-2)=\left(\frac{1}{4k}+3k\right)\left(\frac{20k^2+9}{4k}\right)+(k^2-2)$$

$$=\left(\frac{16k^2+3}{4k}\right)^2 \text{ Hence proved}$$

Some examples are presented below

k	(A,B,C,D)	Property $d(k^2-2)$
1	$\left(1,1,\frac{13}{4},\frac{29}{4}\right)$	$d(-1)$
2	$\left(\frac{1}{2},\frac{1}{2},\frac{49}{8},\frac{89}{8}\right)$	$d(2)$
3	$\left(\frac{1}{3},\frac{1}{3},109,\frac{189}{12}\right)$	$d(7)$
4	$\left(\frac{1}{4},\frac{1}{4},\frac{193}{16},\frac{329}{16}\right)$	$d(14)$
5	$\left(\frac{1}{5},\frac{1}{5},\frac{301}{20},\frac{509}{20}\right)$	$d(23)$
$1+i3$	$\left(\frac{1}{1+i3},\frac{1}{1+i3},\frac{72i-95}{12i+4},\frac{120i-160}{12i+4}\right)$	$d(6i-10)$
$3+\sqrt{2}$	$\left(\frac{1}{3+\sqrt{2}},\frac{1}{3+\sqrt{2}},\frac{157+72\sqrt{2}}{12+4\sqrt{2}},\frac{269+120\sqrt{2}}{12+4\sqrt{2}}\right)$	$d(9+6\sqrt{2})$
$(2+i\sqrt{2})$	$\left(\frac{1}{2+i\sqrt{2}},\frac{1}{2+i\sqrt{2}},\frac{25+48i\sqrt{2}}{8+i4\sqrt{2}},\frac{49+i80\sqrt{2}}{8+i4\sqrt{2}}\right)$	$d(i4\sqrt{2})$

Theorem 4

Let $A=\frac{1}{k}, B=\frac{1}{k}, C=2k^2+3k$ and $D=\frac{2k^3+5k^2+2k+1}{k}$ .Then (A,B,C,D) is a $d(k^2-2)$ –strong rational Diophantine quadruple.

Proof

$$AB+(k^2-2)=\frac{1}{k^2}+(k^2-2)$$

$$=\left(\frac{k^2-1}{k}\right)^2$$

$$AC+(k^2-2)=\frac{1}{k}(2k^2+3k)+(k^2-2)$$

$$=(k+1)^2=BC+(k^2-2) \qquad \text{(since A=B)}$$

$$AD+(k^2-2)=\frac{1}{k}\left(\frac{2k^3+5k^2+2k+1}{k}\right)+(k^2-2)$$

$$=\left[\frac{k^2+k+1}{k}\right]^2=BD+(k^2-2)$$

$$CD+(k^2-2)=(2k^2+3k)\left(\frac{2k^3+5k^2+2k+1}{k}\right)+(k^2-2)$$

$$=(2k^2+4k+1)^2 \text{ Hence proved}$$

Some examples are presented below

k	(A,B,C,D) Property	Property $d(k^2-2)$
1	$(1,1,5,10)$	$d(-1)$
2	$\left(\frac{1}{2},\frac{1}{2},14,\frac{41}{2}\right)$	$d(2)$
3	$\left(\frac{1}{3},\frac{1}{3},27,\frac{106}{3}\right)$	$d(7)$
4	$\left(\frac{1}{4},\frac{1}{4},44,\frac{217}{4}\right)$	$d(14)$
5	$\left(\frac{1}{5},\frac{1}{5},65,\frac{386}{5}\right)$	$d(23)$
$1+i4$	$\left(\frac{1}{1+i4},\frac{1}{1+i4},28i-27,\frac{-(152i+74)}{1+i4}\right)$	$d(8i-17)$
$3+i\sqrt{2}$	$\left(\frac{1}{3+i\sqrt{2}},\frac{1}{3+i\sqrt{2}},23+i15\sqrt{2},60+i82\sqrt{2}\right)$	$d(5+i6\sqrt{2})$
$(2+\sqrt{3})$	$\left(\frac{1}{2+\sqrt{3}},\frac{1}{2+\sqrt{3}},20+11\sqrt{3},92+46\sqrt{3}\right)$	$d(5+4\sqrt{3})$

Theorem 5

Let $A=\frac{1}{2k}, B=\frac{1}{2k}, C=(2\alpha^2-2)k^3+4\alpha k^2$ and $D=\frac{4k^4(\alpha^2-1)+8\alpha k^3+4\alpha k^2+4k+1}{2k}$ .Then (A,B,C,D) is a $d(k^2+1)$ –strong rational Diophantine quadruple.

Proof

$$AB+(k^2+1)=\frac{1}{4k^2}+(k^2+1)$$

$$=\left(\frac{2k^2+1}{2k}\right)^2$$

$$AC+(k^2+1)=\frac{1}{2k}\left((2\alpha^2-2)k^3+4\alpha k^2\right)+(k^2+1)$$

$$=(\alpha k+1)^2=BC+(k^2+1) \qquad \text{(since A=B)}$$

$$AD+(k^2+1)=\frac{1}{k}\left(\frac{4k^4(\alpha^2-1)+8\alpha k^3+4\alpha k^2+4k+1}{2k}\right)+(k^2+1)$$

$$=\left[\frac{2k^2\alpha+2k+1}{2k}\right]^2=BD+(k^2+1)$$

$$CD+(k^2+1)=\{(2\alpha^2-2)k^3+4\alpha k^2\}$$

$$\left(\frac{4k^4(\alpha^2-1)+8\alpha k^3+4\alpha k^2+4k+1}{2k}\right)+(k^2+1)$$

$$=\left(2(\alpha^2-1)k^3+4\alpha k^2+\alpha k+1\right)^2$$ Hence proved

Some examples are presented below

α	k	**(A,B,C,D)**	**Property** $d(k^2+1)$
$\frac{3}{2}$	1	$\left(\frac{1}{2},\frac{1}{2},\frac{17}{2},14\right)$	$d(2)$
2	$\frac{1}{3}$	$\left(\frac{3}{2},\frac{3}{2},\frac{10}{9},\frac{107}{18}\right)$	$d(\frac{10}{9})$
$\frac{2}{3}$	$\frac{1}{2}$	$\left(1,1,\frac{19}{36},\frac{151}{36}\right)$	$d(\frac{5}{4})$
3	5	$\left(\frac{1}{10},\frac{1}{10},2300,\frac{23321}{10}\right)$	$d(26)$

Theorem 6

Let $A=\frac{1}{2k}, B=\frac{1}{2k}, C=2(s^2-1)k^3+4s\alpha k^2+2(\alpha^2-1)k$ and

$D=\frac{4k^4(s^2-1)+8s\alpha k^3+4(\alpha^2+s-1)k^2+4\alpha k+1}{2k}$ .Then (A,B,C,D) is a $d(k^2+1)$ –strong rational Diophantine quadruple.

Proof

$$AB+(k^2+1)=\frac{1}{4k^2}+(k^2+1)$$
$$=\left(\frac{2k^2+1}{2k}\right)^2$$
$$AC+(k^2+1)=\frac{1}{2k}\left(2(s^2-1)k^3+4s\alpha k^2+2(\alpha^2-1)k\right)+(k^2+1)$$
$$=(\alpha+sk)^2=BC+(k^2+1) \qquad \text{(since A=B)}$$
$$AD+(k^2+1)=\frac{1}{2k}\left(\frac{4k^4(s^2-1)+8s\alpha k^3+4(\alpha^2+s-1)k^2+4\alpha k+1}{2k}\right)+(k^2+1)$$
$$=\left[\frac{2k^2s+2k\alpha+1}{2k}\right]^2=BD+(k^2+1)$$
$$CD+(k^2+1)=\{2(s^2-1)k^3+4s\alpha k^2+2(\alpha^2-1)k\}$$
$$\left(\frac{4k^4(s^2-1)+8s\alpha k^3+4(\alpha^2+s-1)k^2+4\alpha k+1}{2k}\right)+(k^2+1)$$
$$=\left(2k^3(s^2-1)+4s\alpha k^2+2(\alpha^2-1)k+ks+\alpha\right)^2 \text{ Hence proved}$$

Some examples are presented below

S	α	k	(A,B,C,D)	Property $d(k^2+1)$
3	$\frac{1}{2}$	1	$\left(\frac{1}{2},\frac{1}{2},\frac{41}{2},28\right)$	$d(2)$
2	1	$\frac{1}{3}$	$\left(\frac{3}{2},\frac{3}{2},\frac{10}{9},\frac{107}{18}\right)$	$d(\frac{10}{9})$
1	2	$\frac{1}{2}$	$(1,1,5,11)$	$d(\frac{5}{4})$
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\left(2,2,\frac{-491}{1152},\frac{2869}{1152}\right)$	$d(\frac{17}{16})$

CONCLUSION

In this paper, we have presented six strong rational Diophantine quadruples. To conclude, one may search for other families of strong and almost strong Diophantine quadruples with suitable property.

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