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On Some Common Fixed Point Theorems for Certain Contractive Mappings in Cone Metric Spaces

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Abstract: The aim of this paper is to establish the generalization of T-Reich and T-Rhoades type mappings on complete cone metric spaces. Huang and Zhang introduce the notion of cone metric spaces. He replaced real number system by ordered Banach space and gave the condition in the setting of cone metric spaces. These authors also described the convergence of sequences in the cone metric spaces and introduce the corresponding notion of completeness. The study of fixed point theorems in such spaces is followed by many researchers. In the present paper this study has been extended to analyse the existence and uniqueness of common fixed points of T-Reich contractive mappings defined on a complete cone metric space (X, d) as well as T-Rhoades contractive mappings. These results generalize and extend the existing fixed point theorems available in the literature.
Keywords: Fixed point, complete cone metric spaces, sequentially convergent.
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1. INTRODUCTION AND PRELIMINARIES

First, we recall some standard notations and definitions in cone metric spaces with some of their properties (see [2].

Definition 1.1. Let E be a real Banach space and P a subset of E. P is called a cone if:
(i) P is closed, non-empty and $P \neq \{0\}$,
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,
(iii) $x \in P$ and $-x \in P \Rightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x \ll y$ if $y - x \in int\, P, int\, P$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,
$0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$.

The least positive number K satisfying the above is called the normal constant of P.

In the following suppose that E is a Banach space, P is a cone in E with $int\, P \neq \emptyset$ and $\leq$ is partial ordering with respect to P.

Definition 1.2. E. a mapping such that $d: X \times X \to E$ a mapping such that
(i) $0 \leq d(x, y)$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called cone metric on X and (X, d) is called cone metric space [2].

Example 1.1. Let $E = R^2, P = \{(x, y) \in E: x, y \geq 0\} \subset R^2, X = R$ and $d: X \times X \to E$ such that $d(x, y) = (|x - y|, \propto|x - y|)$ where $\propto \geq 0$ is a constant. Then (X, d) is a cone metric space **.**

Definition 1.3 [2]. Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) $\{x_n\}$ converges to x if for every $c \in E$ with $0 \ll c$ there is an n_0 such that for all $n \geq n_0, d(x_n, x) \ll c$. We denote this by $lim_{n\to\infty} x_n = x$ or $x_n = x, (n \to \infty)$.

(ii) If for any $c \in E$ with $0 \ll c$, there is an n_0 such that for all $n \geq n_0$, $d(x_n, x) \ll c$. Then $\{x_n\}$ is a Cauchy sequence in X. (X, d) is called a complete cone metric space, if Cauchy sequence in X is convergent in X.

Lemma 1.1[2]. Let (X, d) be a cone metric space, $P \subset E$ a normal cone with normal constant K Let $\{x_n\}$,$\{y_n\}$ be a sequence in X and $x, y \in X$.

(i) $\{x_n\}$ converges to x if and only if $lim_{n\to\infty} d(x_n, x) = 0$;

(ii) If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y then $x = y$. That is the limit of $\{x_n\}$ is unique.

(iii) If $\{x_n\}$ converges to x, then $\{x_n\}$ is Cauchy sequence.

(iv) $\{x_n\}$ is a Cauchy sequence if for $lim_{n\to\infty} d(x_n, x_m) = 0$;

if $x_n \to x$ and $y_n \to y (n \to \infty)$ then $d(x_n, y_n) \to d(x, y)$.

Definition 1.4 [5]. Let (X, d) be a cone metric space, P a normal cone with normal constant K and $T: X \to X$. Then

(i) T is said to be continuous if $lim_{n\to\infty} x_n = x$ implies that $lim_{n\to\infty} T(x_n) = T(x)$, for all $\{x_n\}$ in X.

(ii) T is said to be sub-sequentially convergent, if we have for every sequence $\{y_n\}$ that $T\{y_n\}$ is convergent, implies $\{y_n\}$ has a convergent subsequence.

(iii) T is said to be sequentially convergent, if for every sequence $\{y_n\}$, $T\{y_n\}$ is convergent, and then $\{y_n\}$ also is convergent.

Definition 1.5 [4,5]. Let (X, d) be a cone metric space and $T, S: X \to X$ two functions,

(i) A mapping S is said to be T-Reich contraction if there is $a + b + c < 1$ such that

$d(TSx, TSy) \leq ad(Tx, TSx) + bd(Ty, TSy) + cd(Tx, Ty)$ for all $x, y \in X$ and $a, b, c \geq 0$.

(ii) A mapping S is said to be T-Rhoades contraction if there is $a + b + c < 1$ such that

$d(TSx, TSy) \leq ad(Tx, TSy) + bd(Ty, TSx) + cd(Tx, Ty)$ for all $x, y \in X$ and $a, b, c \geq 0$.

2. MAIN RESULTS

Theorem 2.1. Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K, in addition let $T: X \to X$ be a one to one continuous function and $R, S: X \to X$ be a pair of T-Reich contraction. Then

(1) For every $x_0 \in X$,

$lim_{n\to\infty} d(TR^{2n+1}x_0, TR^{2n+2}x_0) = 0$ and
$lim_{n\to\infty} d(TS^{2n+2}x_0, TS^{2n+3}x_0) = 0$;

(2) There is $\vartheta \in X$ such that
$lim_{n\to\infty} TR^{2n+1}x_0 = \vartheta = lim_{n\to\infty} TS^{2n+2}x_0$;

(3) If T is subsequentially convergent, then $\{R^{2n+1}x_0\}$ and $\{S^{2n+2}x_0\}$ have a convergent subsequences;

(4) There is unique common fixed point $u \in X$ such that $Ru = u = Su$;

(5) If T is a sequentially convergent, then for each $x_0 \in X$ the iterate sequences $\{R^{2n+1}x_0\}$ and $\{S^{2n+2}x_0\}$ converge to u.

Proof: Let x_0 be any arbitrary point in X. We define the iterate sequences $\{x_{2n+1}\}$ and $\{x_{2n+2}\}$ by

$$x_{2n+2} = Rx_{2n+1} = R^{2n+1}x_0 \text{ and}$$
$$x_{2n+3} = Sx_{2n+2} = S^{2n+2}x_0;$$

Since R and S are pair of T-Reich contraction, we have

$$\begin{aligned} d(Tx_{2n+1}, Tx_{2n+2}) &= d(TRx_{2n}, TRx_{2n+1}) \\ &\leq ad(Tx_{2n}, TRx_{2n}) + bd(Tx_{2n+1}, TRx_{2n+1}) \\ &\quad + cd(Tx_{2n}, Tx_{2n+1}) \\ &\leq ad(Tx_{2n}, Tx_{2n+1}) + bd(Tx_{2n+1}, Tx_{2n+2}) \\ &\quad + cd(Tx_{2n}, Tx_{2n+1}) \end{aligned}$$

$$d(Tx_{2n+1}, Tx_{2n+2}) \leq \left(\frac{a + c}{1 - b}\right) d(Tx_{2n}, Tx_{2n+1})$$

$$d(Tx_{2n+2}, Tx_{2n+3}) \leq \left(\frac{a' + c'}{1 - b'}\right) d(Tx_{2n+1}, Tx_{2n+2})$$

We can conclude, by repeating the same argument, that

$d(TR^{2n+1}x_0, TR^{2n+2}x_0) \leq \left(\frac{a+c}{1-b}\right)^{2n+1} d(Tx_0, TRx_0)$ **(2.1)**

And $d(TS^{2n+2}x_0, TS^{2n+3}x_0) \leq \left(\frac{a'+c'}{1-b'}\right)^{2n+2} d(Tx_0, TSx_0)$ **(2.2)**

From (2.1) we have

$\|d(TR^{2n+1}x_0, TR^{2n+2}x_0)\| \leq \left(\frac{a+c}{1-b}\right)^{2n+1} K\|d(Tx_0, TRx_0)\|$

Where K is the normal constant of E. By above inequality we get

$lim_{n\to\infty}\|d(TR^{2n+1}x_0, TR^{2n+2}x_0)\| = 0$ **(2.3)**

By inequality (2.1), for every $m, n, \in N$ with $m > n$, we have

$$d(Tx_{2n+1}, Tx_{2m+1}) \leq d(Tx_{2n+1}, Tx_{2n+2}) + - - - + d(Tx_{2m}, Tx_{2m+1})$$
$$\leq \left[\left(\frac{a+c}{1-b}\right)^{2n+1} + - - - - - + \left(\frac{a+c}{1-b}\right)^{2m}\right] d(Tx_o, TRx_0)$$
$$= \left(\frac{a+c}{1-b}\right)^{2n+1} \times \frac{1}{1-\frac{a+c}{1-b}} d(Tx_0, TRx_0)$$

$d(TR^{2n+1}x_0, TR^{2n+2}x_0) \leq \left(\frac{a+c}{1-b}\right)^{2n+1} \frac{1}{1-\frac{a+c}{1-b}} d(Tx_0, TRx_0)$ **(2.4)**

From (2.4) we have,

$$\|d(TR^{2n+1}x_0, TR^{2m+1}x_0)\| \leq \left(\frac{a+c}{1-b}\right)^{2n+1} \times \frac{K}{1-\frac{a+c}{1-b}} d(Tx_0, TRx_0)$$

Where K is the normal constant of E. Taking limit and by $\frac{a+c}{1-b} < 1$, we obtain

$lim_{n\to\infty}\|d(TR^{2n+1}x_0, TR^{2m+1}x_0)\| = 0.$

In this way, we have

$lim_{n\to\infty} d(TR^{2n+1}x_0, TR^{2m+1}x_0) = 0$, which implies that $\{TR^{2n+1}x_0\}$ is a Cauchy sequence in X. Since X is a complete cone metric space, than there is $\vartheta \in X$ such that

$lim_{n\to\infty} TR^{2n+1}x_0 = \vartheta$ **(2.5)**

Now, if T is subsequentially convergent, $\{R^{2n+1}x_0\}$ has a convergent subsequence. So there are $u \in X$ and $\{x_{(2n+1)_i}\}$ such that

$lim_{n\to\infty} R^{(2n+1)_i}x_0 = u$ **(2.6)**

Since T is continuous and by (2.6) we obtain

$lim_{n\to\infty} TR^{(2n+1)_i}x_0 = Tu$ **(2.7)**

By (2.5) and (2.7) we conclude that

$Tu = \vartheta$ **(2.8)**

On the other hand,

$$d(TRu, Tu) \leq d\left(TRu, TR^{(2n+1)_i+1}(x_0)\right) + d(TR^{(2n+1)_i}x_0, TR^{(2n+1)_i+1}x_0)$$
$$+d(TR^{(2n+1)_i+1}\, x_0, \text{Tu})$$
$$\leq ad(Tu, TRu) + bd(TR^{(2n+1)_i-1}x_0, TR^{(2n+1)_i-1}x_0)$$
$$+cd\left(Tu, TR^{(2n+1)_i-1}x_0\right) + \left(\frac{a+c}{1-b}\right)^{(2n+1)_i} d(Tx_0, TRx_0)$$
$$+d(TR^{(2n+1)_i+1}x_0, Tu)$$

Therefore,

$$(1-a)d(TRu, Tu) \leq bd(TR^{(2n+1)_i-1}x_0, TR^{(2n+1)_i-1}x_0)$$
$$+cd(Tu, TR^{(2n+1)_i+1}x_0)$$
$$+\left(\frac{a+c}{1-b}\right)^{(2n+1)_i} d(Tx_0, TRx_0)$$
$$+d(TR^{(2n+1)_i+1}x_0, Tu)$$
$$d(TRu, Tu) \leq (\frac{b}{1-a}) d(TR^{(2n+1)_i-1}x_0, TR^{(2n+1)_i-1}x_0)$$

$$+(\tfrac{c}{1-a})d(Tu,TR^{(2n+1)_i+1}x_0)$$
$$+(\tfrac{1}{1-a})\left(\tfrac{a+c}{1-b}\right)^{(2n+1)_i} d(Tx_0,TRx_0)$$
$$+(\tfrac{1}{1-a})d(TR^{(2n+1)_i+1},Tu)$$
$$\|d(TRu,Tu)\| \le \frac{bk}{1-a}\|d(TR^{(2n+1)_i-1}x_0,TR^{(2n+1)_i-1}x_0)\|$$
$$+\tfrac{c}{1-a}K\|d(Tu,TR^{(2n+1)_i-1}x_0)\|$$
$$+(\tfrac{1}{1-a})\left(\tfrac{a+c}{1-b}\right)^{(2n+1)_i} K\|d(Tx_0,TRx_0)\|$$
$$+(\tfrac{1}{1-a})K\|d(TR^{(2n+1)_i+1},Tu)\| \to 0 \text{ as } i\to\infty.$$
$$d(TRu,TR\vartheta)a[d(Tu,TRu)]+b[d(T\vartheta,TR\vartheta)]+c[d(Tu,T\vartheta)].$$

Where K is the normal constant of X.Hence $d(TRu,Tu)=0$, which implies that $d(TRu,Tu)=0$, $TRu=Tu$. Since T is one to one, we have $Ru=u$. Hence R has a fixed point. Because R is T-Reich contraction, we have

If ϑ is another fixed point of R, then from the injectivity of T, we get $Ru=R\vartheta$.

Hence fixed point is unique. Finally, if T is sequentially convergent, by replacing (2n+1) for $((2n+1)_i)$, we conclude that

$lim_{n\to\infty} R^{2n+1}x_0 = u.$

This shows that $(R^{2n+1}x_0)$ converges to the fixed point of R.

Similarly, it can be established that $(S^{2n+2}x_0)$ converges to the fixed point of S.

That is $lim_{n\to\infty} R^{2n+1}x_0 = u = lim_{n\to\infty} S^{2n+2}x_0$.

Theorem 2.2. Let (X,d) be a complete cone metric space, P be a normal cone with normal constant K, in addition $T: X\to X$ be a one to one continuous function and $R,S: X\to X$ be a pair of T-Rhoades contraction then (1), (2), (3), (4) and (5) of Theorem 2.1 hold.

Proof. Let x_0 be an arbitrary point in X. We define the iterate sequences $\{x_{2n+1}\}$ and $\{x_{2n+2}\}$ by

$$x_{2n+2}=Rx_{2n+1}=R^{2n+1}x_0 \text{ and}$$
$$x_{2n+3}=Sx_{2n+2}=S^{2n+2}x_0;$$

Since R and S are pair of T-Rhoades contraction, we have

$$d(Tx_{2n+1},Tx_{2n+2}) = d(TRx_{2n},TRx_{2n+1})$$
$$\le ad(Tx_{2n},TRx_{2n+1})+bd(Tx_{2n+1},TRx_{2n})$$
$$+cd(Tx_{2n},Tx_{2n+1})$$
$$\le ad(Tx_{2n},Tx_{2n+2})+bd(Tx_{2n+1},Tx_{2n+1})+cd(Tx_{2n},Tx_{2n+1})$$
$$\le a[d(Tx_{2n},Tx_{2n+1})+d(Tx_{2n+1},Tx_{2n+2})] \quad +cd(Tx_{2n},Tx_{2n+1})$$
$$d(Tx_{2n+1},Tx_{2n+2}) \le \left(\frac{a+c}{1-b}\right)d(Tx_{2n},Tx_{2n+1})$$
$$d(Tx_{2n+2},Tx_{2n+3}) \le \left(\frac{a'+c'}{1-a'}\right)d(Tx_{2n+1},Tx_{2n+2})$$
$$d(Tx_{2n+1},Tx_{2n+2}) \le \left(\frac{a+c}{1-a}\right)d(Tx_{2n},Tx_{2n+1})$$
$$\le hd(TRx_{2n},TRx_{2n+1}).$$

Where $h=\frac{a+c}{1-a}$. Recursively, we obtain

$$d(TRx_{2n+1},TRx_{2n+2}) \le h^{2n+1}\, d(TRx_0,TRx_1) \qquad \textbf{(2.9)}$$

Therefore,

$$\|d(TRx_{2n+1},TRx_{2n+2})\| \le h^{2n+1}K\|d(TRx_0,TRx_1)\|$$

Where K is the normal constant of X.
Hence $lim_{n\to\infty}\|d(TRx_{2n+1},TRx_{2n+2})\|=0$,

This implies that

$$lim_{n\to\infty} d(TR^{2n+1}x_0, TR^{2n+2}x_0) = 0$$

Similarly, we have

$$lim_{n\to\infty} d(TS^{2n+2}x_0, TS^{2n+3}x_0) = 0$$

By (2.9), for every $m, n \in N$ with $n > m$, we have $d(TRx_{2m+1}, TRx_{2n+1}) \leq d(TRx_{2n+1}, TRx_{2n+2}) + ----$ $+d(TRx_{2m}, TRx_{2m+1})$

$$\leq [h^{2n} + h^{2n-1} \pm - \mp h^{2m+1}] d(TRx_0, TRx_1)$$

$$\leq \frac{h^{2m+1}}{1-h} d(TRx_0, TRx_1)$$

Taking norm we get

$\|d(TRx_{2m+1}, TRx_{2n+1})\| \leq \frac{h^{2m+1}}{1-h} K \|d(TRx_0, TRx_1)\|.$

Consequently, we have

$$lim_{n,m\to\infty} d(TRx_{2n+1}, TRx_{2m+1}) = 0.$$

Hence $\{TR^{2n+1}x_0\}$ is a Cauchy sequence in X is complete cone metric space, there is $\vartheta \in X$ such that

$$lim_{n\to\infty} TR^{2n+1}x_0 = \vartheta.$$

Similarly we can prove that $lim_{n\to\infty} TS^{2n+2}x_0 = \vartheta$ The rest of the proof is similar to the proof of Theorem 2.1.

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