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Ascertain Subclasses of Meromorphically Multivalent Functions with Negative Coefficient Associated with Linear Operator

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Abstract: In this paper, we introduce the subclasses $A^{n}_{\lambda,P}(a,b,c;\alpha,A,B)$ and $A^{*n}_{\lambda,P}(a,b,c;\alpha,A,B)$ of meromorphic multivalent functions in the punctured unit disk $U^{*}=\{z\in£ : 0<|z|<1\}$ by using a differential operator $h^{n}_{\lambda,p}(a,b;c,z)f(z)$. We obtain coefficient estimates, distortion theorem, radius of convexity and closure Theorems for the class. $A^{*n}_{\lambda,P}(a,b,c;\alpha,A,B)$.

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INTRODUCTION

Let $\sum_p$ denote the class of functions of the form

$$f(z)=\frac{1}{z^{p}}+\sum_{k=0}^{\infty}a_{k+p}z^{k+p}\ \left(p\in \mathrm{N}=\{1,2,3...\}\right), \tag{1.1}$$

which are analytic in the punctured unit disk $U^{*}=\{z\in£ : 0<|z|<1\}=U/\{0\}$.

Also let Ω_p denote the subclass of $\sum_p$ of meromorphicmultivalent functions in U^{*}, which have the power series representation

$$f(z)=\frac{1}{z^{p}}-\sum_{k=0}^{\infty}a_{k+p}z^{k+p}\quad (a_{k+p}\geq 0)\ . \tag{1.2}$$

A function $f(z)\in\sum_p$ is said to be p -valent meromorphically starlike of order α , if and only if

$$\mathrm{Re}\left\{-\frac{zf'(z)}{f(z)}\right\}>\alpha \qquad (z\in U^{*})\ , \tag{1.3}$$

for some $\alpha(0\leq\alpha<p)$. We denoted the class of all meromorophic p -valent starlike functions of order α by $\sum_p(\alpha)$. Further a function $f(z)$ in $\sum_p$ is said to bemeromorophic p -valent convex of order α if and only if

$$\mathrm{Re}\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\alpha \qquad (z\in U^{*}), \tag{1.4}$$

for some $\alpha(0\leq\alpha<p)$. We denote the class of all meromorophic p -valent convex functions of order α by $K_p(\alpha)$. The classes $\sum_p(\alpha)$ and $K_p(\alpha)$ and various other subclasses of $\sum_p$ have been studied rather extensively by Aouf *et.al.* ([1] , [3] and [5]), Joshi and Srivastava [6], Kulkarni *et. al.* [7], Owa *et.al.* [10], and others.
For $\alpha=0$, we obtain the class $\sum(p)$ and $K\left(p\right)$ of meromorophic p-valent starlike and convex functions with respect to the origin.

Denote by $\sum_p^*(\alpha)$ and $K_p^*(\alpha)$ the classes obtained by considering intersection, respectively, of the classes $\sum_p(\alpha)$ and $K_p(\alpha)$ with Ω_p, i.e.

$$\begin{aligned} \sum\nolimits_p^*(\alpha) &= \sum\nolimits_p(\alpha) \cap \Omega_p \quad ; \quad (0 \le \alpha < p) \\ K_p^*(\alpha) &= K_p(\alpha) \cap \Omega_p \quad ; \quad (0 \le \alpha < p) \end{aligned} \tag{1.5}$$

The function $f(z)$ is said to be subordinate to $F(z)$, if there exists a function $w(z)$ analytic in U with $w(0)=0$ and $|w(z)|<1$ $(z \in U)$, such that $f(z)=F(w(z))$. In such a case we write $f(z) \prec F(z)$. In particular, if F is univalent, then $f(z) \prec F(z)$ if and only if $f(0)=F(0)$ and $f(U) \subset F(U)$.

For $f(z) \in \sum_p$ given by (1.1) and $g(z) \in \sum_p$ given by

$$g(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} b_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1,2,3\ldots\}), \tag{1.6}$$

the Hadamard product (or convolution) of f and g is denoted by $(f*g)(z)$ and defined by

$$(f*g)(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_{k+p} b_{k+p} z^{k+p} \tag{1.7}$$

For the function $f(z) \in \sum_p$, Aouf [2] define the following differential operator

$$S_{\lambda,p}^0 f(z) = f(z)$$

$$S_{\lambda,p}^1 f(z) = (1-\lambda) f(z) + \frac{\lambda}{p} z f'(z) + \frac{2\lambda}{z^p}$$

$$= \frac{1}{z^p} + \sum_{k=0}^{\infty} \left(1 + \frac{\lambda k}{p}\right) a_{p+k} z^{p+k} = S_{\lambda,p} f(z). \qquad (\lambda \ge 0, p \in N).$$

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$$S_{\lambda,p}^2 f(z) = S_{\lambda,p}(D_{\lambda,p}^1 f(z)).$$

$$\begin{aligned} S_{\lambda,p}^n f(z) &= S_{\lambda,p}(S_{\lambda,p}^{n-1} f(z)) \\ &= (1-\lambda) S_{\lambda,p}^{n-1} f(z) + \frac{\lambda}{p} z \left(S_{\lambda,p}^{n-1} f(z)\right)' + \frac{2\lambda}{z^p} \quad (\lambda \ge 0; n, p \in N). \end{aligned} \tag{1.8}$$

It can be easily seen that

$$S_{\lambda,p}^n f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \left(1 + \frac{k\lambda}{p}\right)^n a_{p+k} z^{p+k} \quad (n \in N_0 = N \cup \{0\}; p \in N). \tag{1.9}$$

For positive numbers a, b and c, define the operator $I_p(a,b;c,z)$ by

$$I_p(a,b;c,z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^{p+k}. \tag{1.10}$$

With the aid of the operator $S_{\lambda,p}^n f(z)$ defined by (1.9) and the operator $I_p(a,b;c,z)$, defined by (1.10) we define the operator $h_{\lambda,p}^n(a,b;c,z) f(z)$ in terms of the hadmered product or (convolution) by

$$h_{\lambda,p}^n(a,b;c,z) f(z) = S_{\lambda,p}^n f(z) * I_p(a,b;c,z), \tag{1.11}$$

which can be written for $f(z)$ defined by (1.1) as

$$h_{\lambda,p}^{n}(a,b;c,z)f(z)=\frac{1}{z^{p}}+\sum_{k=0}^{\infty}Q_{k,\lambda}^{n}(a,b;c)a_{p+k}z^{p+k} \quad . \tag{1.12}$$

for $n\in N$, $\lambda\geq 0$,
where

$$Q_{k,\lambda}^{n}(a,b;c)=(1+\frac{k\lambda}{p})^{n}\frac{(a)_{k}(b)_{k}}{(c)_{k}(1)_{k}} \tag{1.13}$$

With the aid of the differential operator $h_{\lambda,p}^{n}(a,b;c,z)f(z)$ we define the following subclasses of multivalent and meromorphic functions.

Definition 1.A function $f(z)\in\sum_{p}$ defined by (1.1) is said to be in the class $A_{\lambda,p}^{n}(a,b,c;\alpha,A,B)$ if it satisfies the following subordination condition:

$$1+\frac{z(h_{\lambda,p}^{n}(a,b;c,z)f(z))''}{(h_{\lambda,p}^{n}(a,b;c,z)f(z))'}\ \mathrm{p}\ \frac{p+\left[pB+(A-B)(p-\alpha)\right]z}{1+Bz}\quad (z\in U^{*}) \qquad (\qquad 1.14)$$

or , equivalently, if the following inequality holds true:

$$\left|\frac{1+\dfrac{(z(h_{\lambda,p}^{n}(a,b;c,z)f(z))''}{(h_{\lambda,p}^{n}(a,b;c,z)f(z))'}+p}{B(1+\dfrac{(z(h_{\lambda,p}^{n}(a,b;c,z)f(z))'}{(h_{\lambda,p}^{n}(a,b;c,z)f(z)}}\right|\ \mathrm{p}\ 1 \qquad (z\in U^{*}) \tag{1.15}$$

Also let $A_{\lambda,P}^{*n}(a,b,c;\alpha,A,B)=A_{\lambda,P}^{n}(a,b,c;\alpha,A,B)\ \mathrm{I}\ \Omega_{p}$

$$(0\leq\alpha<P;-1\leq A<B\leq 1;0<B\leq 1;p\in N;n\in N_{0};\lambda\geq 0;\ \delta\geq 0)$$

It may be noted that for suitable choice of δ,A,B,n,p,λ and α . the class $A_{\lambda,p}^{*n}(a,b,c;\alpha,A,B)$ extends several classes of analytic and p -valent meromorphic functions such that Aouf and Shammaky[4], Srivastava et al. [11] , Liu. and Srivastava ([8], [9]) and Uralegaddi and Ganigi [12].

Basic properties of the class $A_{\lambda,p}^{*n}(a,b,c;\alpha,A,B)$**.**

We first determine a necessary and sufficient condition for a function $f(z)\in\Omega_{p}$ of the form (1.2) to be in the class $A_{\lambda,p}^{*}(a,b,c;\alpha,A,B)$

Theorem 1.*Let the function* $f(z)\in\Omega_{p}$ *defined by (1.2), the* $f(z)\in A_{\lambda,p}^{*n}(a,b,c;\alpha,A,B)$ *if and only if*

$$\sum_{k=0}^{\infty}(k+p)Q_{k,\lambda}^{n}(a,b;c)M_{k}(\alpha,A,B,P)a_{k+p}\leq p(B-A)(p-\alpha) \tag{2.1}$$

$$(0\leq\alpha<P;A+B\geq 0;\ -1\leq A<B\leq 1;0<B\leq 1;p\in N;n\in N_{0};\lambda\geq 0;\delta\geq 0)$$

where

$$M_{K}(\alpha,A,B,P)=\left[\left(k+p\right)(B+1)+p(A+1)+(B-A)\alpha\right], \tag{2.2}$$

and $Q_{k,\lambda}^{n}(a,b;c)$ *is given by (1.13).*

Proof. Suppose that the function $f(z)\in\Omega_{p}$ defined by (1.2) be in the class $A_{\lambda,p}^{*n}(a,b,c;\alpha,A,B)$, then from (1.15) we have

$$\left|\frac{(z(h_{\lambda,p}^{n}(a,b;c,z)f(z))''+(1+p)(h_{\lambda,p}^{n}(a,b;c,z)f(z))'}{B(h_{\lambda,p}^{n}(a,b;c,z))'+z(h_{\lambda,p}^{n}(a,b;c,z))''+\left[pB+(A-B)(p-\alpha\right](h_{\lambda,p}^{n}(a,b;c,z)f(z))'}\right|=$$

$$\left|\frac{\left\{-\sum_{k=0}^{\infty}(k+p)(k+2p)Q_{k,\lambda}^{n}(a,b;c)a_{k+p}z^{k+2p}\right\}n}{p(B-A)(p-\alpha)-\sum_{k=0}^{\infty}(k+p)Q_{k,\lambda}^{n}(a,b;c)\left[(k+p)B+(B-A)\alpha+AP\right]a_{k+p}z^{k+2p}}\right|$$

$$<1 \quad (z\in U). \tag{2.3}$$

Since $\left|\operatorname{Re}\{z\}\right|\le|z|$ for any z, choosing z to be real and letting $z\to1^{-}$ through real value, then (2.3) yield

$$\sum_{k=0}^{\infty}(k+p)(k+2p)Q_{k,\lambda}^{n}(a,b;c)a_{k+p}$$

$$\le p(B-A)(p-\alpha)-\sum_{k=0}^{\infty}(k+p)Q_{k,\lambda}^{n}(a,b;c)\left[(k+p)B+(B-A)\alpha+AP\right]A_{K+P}, \tag{2.4}$$

which leads us immediate to the coefficient inequality (2.1).

Next in order to prove the converse we assume that the inequality (2.1) holds true, then we observe that

$$\left|\frac{(z(h_{\lambda,p}^{n}(a,b;c,z)f(z))''+(1+p)(h_{\lambda,p}^{n}(a,bc,z)f(z))'}{B(h_{\lambda,p}^{n}(a,b;c,z)f(z))'+z(h_{\lambda,p}^{n}(a,b;c,z)f(z))'')+\left[pB+(A-B)(P-\alpha)\right](h_{\lambda,p}^{n}(a,b;c,z)f(z))'}\right|$$

$$\le\frac{\sum_{k=0}^{\infty}(k+p)(k+2p)Q_{k,\lambda}^{n}(a,b;c)a_{k+p}}{p(B-A)(P-\alpha)-\sum_{K=0}^{\infty}(k+p)Q_{k,\lambda}^{n}(a,b;c)\left[(k+p)B+(B-A)\alpha+AP\right]a_{k+p}}$$

$$<1 \quad (z\in U). \tag{2.5}$$

Hence by maximum modulus theorem, we have $f(z)\in A_{\lambda,p}^{*n}(a,b,c;\alpha,A,B)$. This completes the proof of Theorem.

Corollary 1. *Let the function $f(z)\in\Omega_p$ defined by (1.2), if $f(z)\in A_{\lambda,p}^{*n}(a,b,c;\alpha,A,B)$ then*

$$a_{k+p}\le\frac{p(B-A)(P-\alpha)}{(K+P)Q_{k,\lambda}^{n}(a,b;c)M_k(\alpha,A,p)}z^{k+p}(k\ge0,p\in N).. \tag{2.6}$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^{-p}-\frac{p(B-A)(P-\alpha)}{(k+p)Q_{k,\lambda}^{n}(a,b;c)M_k(\alpha,A,B,P)}z^{k+p}\ (k\ge0,p\in N) \tag{2.7}$$

Next we prove the following distortion and growth properties for the class. $A_{\lambda,p}^{*n}(a,b,c;\alpha,A,B)$.

Theorem 2. *If a function $f(z)\in\Omega_p$ defined by (1.2) is in the class $A_{\lambda,p}^{*n}(a,b,c;\alpha,A,B)$, then*

$$\left[\frac{(p+m-1)!}{(p-1)!}-\frac{p!(B-A)(P-\alpha)}{(p-m)!M_{\text{o}}(\alpha,A,B,P)}r^{2p}\right]r^{-p-m}$$

$$\le\left|f^{m}(z)\right|\le\left[\frac{(p+m-1)!}{(p-1)!}+\frac{p!(B-A)(P-\alpha)}{(p-m)!M_{\text{o}}(\alpha,A,B,p)}r^{2p}\right]r^{-p-m} \tag{2.8}$$

$$(0<|z|=r<1,0\le m<p),$$

where the result is sharp for the function $f(z)$ given by

$$f(z)=z^{-p}-\frac{(B-A)(p-\alpha)}{M_{\text{o}}(\alpha,A,B,p)}z^{p}(p\in N)., \tag{2.9}$$

and

$$M_o(\alpha,A,B,P)=\left[p(B+1)+p(A+1)+(B-A)\alpha\right].$$

$$(0\le\alpha<P;A+B\ge 0;\ -1\le A<B\le 1;0<B\le 1;p\in N;n\in N_0;\lambda\ge 0;\delta\ge 0)$$

Proof. For $f(z)\in A^{*n}_{\lambda,p}(a,b,c;\alpha,A,B)$ we find from Theorem 1, that

$$pQ^n_{0,\lambda}(a,b;c)M_0(\alpha,A,B,P)\sum_{K=0}^{\infty}a_{k+p}\le\sum_{k=0}^{\infty}(k+p)Q^n_{k,\lambda}(a,b;c)M_K(\alpha,A,B,P)a_{k+p}$$
$$\le p(B-A)(P-\alpha),$$

or

$$\sum_{k=0}^{\infty}a_{k+1}\le\frac{(B-A)(P-\alpha)}{Q^n_{0,\lambda}(a,b;c)M_0(\alpha,A,B,P)}=\frac{(B-A)(P-\alpha)}{M_0(\alpha,A,B,P)},\tag{2.10}$$

Now by differentiating $f(z)$ in (1.2) m times, we have

$$f^m(z)=(-1)^m\frac{(p+m-1)!}{(p-1)!}z^{-p-m}-\sum_{k=0}^{\infty}\frac{(k+p)!}{(k+p-m)!}\left|a_{k+p}\right|z^{k+p-m}\quad,\tag{2.11}$$
$$(m\in N_0,P\in N,m<P)$$

Thus, for $0\le|z|=r<1$,

$$\left|f^m(z)\right|=\left|(-1)^m\frac{(p+m-1)!}{(p-1)!}z^{-p-m}-\sum_{k=0}^{\infty}\frac{(k+p)!}{(k+p-m)!}a_{k+p}z^{k+p-m}\right|$$
$$\le\frac{(p+m-1)!}{(p-1)!}r^{-p-m}+\sum_{k=0}^{\infty}\frac{(k+p)!}{(k+p-m)!}a_{k+p}r^{k+p-m}$$
$$\le\frac{(p+m-1)!}{(p-1)!}r^{-p-m}+\frac{p!}{(p-m)!}r^{p-m}\sum_{k=0}^{\infty}a_{k+p}$$
$$\le\frac{(P+M-1)!}{(P-1)!}r^{-p-m}+\frac{p!}{(p-m)!}\frac{(B-A)(P-\alpha)}{M_0(\alpha,A,B,P)}r^{p-m},,$$

similarly

$$\ge\frac{(P+M-1)!}{(P-1)!}r^{-p-m}+\frac{p!}{(p-m)!}\frac{(B-A)(P-\alpha)}{M_0(\alpha,A,B,P)}r^{p-m},.$$

The sharpness of each the inequality in (2.8) satisfies by the function $f(z)$ given by (2.9).

Next we determine the radii of meromorphically p-valent starlikeness and convexity of order $\gamma(0\le\gamma<p)$ for functions in the class. $A^{*n}_{\lambda,p}(a,b,c;\alpha,A,B)$

Theorem 3. *If a function $f(z)\in\Omega_p$ defined by (1.2) is in the class, $A^{*n}_{\lambda,p}(a,b,c;\alpha,A,B)$ then*

(i) $f(z)$ is meromorphically p-valent starlike of order $\gamma(0\le\gamma<p)$ in $|z|<r_1$, where

$$r_1=\inf_{k\ge 0}\left\{Q^n_{k,\lambda}(a,b;c)\frac{(k+p)(p-\gamma)M_x(\alpha,A,B,P)}{P(k+p+\gamma)(B-A(p-\alpha)}\right\}^{\frac{1}{k+2p}}\tag{2.12}$$

(ii) $f(z)$ is meromorphically p-valent convex of order $\gamma(0\le\gamma<p)$ in $|z|<r_2$, where

$$r_2=\inf_{k\ge 0}\left\{Q^n_{k,\lambda}(a,b;c)\frac{(p-\gamma)M_k(\alpha,A,B,P)}{(K+P+\gamma)(B-A)(p-\alpha)}\right\}^{\frac{1}{k+2p}}.\tag{2.13}$$

$$(0\le\alpha<P;A+B\ge 0;\ -1\le A<B\le 1;0<B\le 1;p\in N;n\in N_0;\lambda\ge 0;\delta\ge 0).$$

The result are sharp.

Proof.(i)from (1.2),we easily get

$$\left|\frac{z\dfrac{f'(z)}{f(z)}+p}{z\dfrac{f'(z)}{f(z)}-p+2\gamma}\right| \le \frac{\sum_{k=0}^{\infty}(k+2p)a_{k+p}|z|^{k+2p}}{2(\gamma+p)-\sum_{k=0}^{\infty}(k+2\gamma)a_{k+p}|z|^{k+2p}}.$$

Thus, we have the desired inequity:

$$\left|\frac{z\dfrac{f'(z)}{f(z)}+p}{z\dfrac{f'(z)}{f(z)}-p+2\gamma}\right| \le 1 \quad (0\le\gamma<p, p\in N), \tag{2.14}$$

if

$$\sum_{k=0}^{\infty}\frac{(k+p+\gamma)}{(p-\gamma)}a_{k+p}|z|^{k+2p} \le 1. \tag{2.15}$$

Hence , by Theorem 1, (2.15) will be true if

$$\frac{(k+p+\gamma)}{(p-\gamma)}|z|^{k+2p} \le \frac{(k+p)Q_{k,\lambda}^{n}(a,b;c)M_k(\alpha,A,B,P)}{p(B-A)(P-\alpha)}(K\ge 0, P\in N) \tag{2.16}$$

The inequality (2.16) leads us immediately to $|z|<r_1$, where r_1 is given by (2.12).

(ii) In order to prove the second assertion of the Theorem we find from (1.2) that

$$\left|\frac{1+z\dfrac{f''(z)}{f'(z)}+p}{1+z\dfrac{f''(z)}{f'(z)}-p+2\gamma}\right| \le \frac{\sum_{k=0}^{\infty}(k+p)(k+2p)a_{k+p}|z|^{k+2p}}{2p(p-\gamma)-\sum_{k=0}^{\infty}(k+2\gamma)a_{k+p}|z|^{k+2p}}.$$

Thus we have the desired inequity:

$$\left|\frac{1+z\dfrac{f''(z)}{f'(z)}+p}{1+z\dfrac{f''(z)}{f'(z)}-p+2\gamma}\right| \le 1 \qquad (0\le\gamma<p, p\in N), \tag{2.17}$$

If

$$\sum_{k=0}^{\infty}\frac{(k+p)(k+p+\gamma)}{p(p-\gamma)}a_{k+p}|z|^{k+2p} \le 1. \tag{2.18}$$

Hence , by Theorem 1, (2.18) will be true if

$$\frac{(k+p)(k+p+\gamma)}{p(p-\gamma)}|z|^{k+2p} \le \frac{(k+p)Q_{k,\lambda}^{n}(a,b;c)M_k(\alpha,A,B,P)}{P(B-A)(p-\alpha)}(k\ge 0, p\in N), \tag{2.19}$$

the inequality (2.19) leads us immediately to $|z|<r_2$ where r_2 is given by (2.13).

Each of these result is sharp for the function $f(z)$ given by (2.9).

Next we prove closure Theorems for the class. $A_{\lambda,p}^{*n}(a,b,c;\alpha,A,B)$

Theorem 4. *Let*

$$f_{-1}=\frac{1}{z^{p}} \tag{2.20}$$

and

$$f_{p+k}(z)=\frac{1}{z^{p}}-\frac{p(B-A)(P-\alpha)}{(P+K)Q_{k,\lambda}^{n}(a,b;c)M_k(\alpha,A,B,P)}Z^{P+K} \quad (k\ge 0; p\in N; n\in N_0) \tag{2.21}$$

Then $f(z)$ *in the class* $A_{\lambda,p}^{*}(a,b,c;\alpha,A,B)$ *if and only if it can expressed in the form*

$$f(z)=\sum_{k=-1}^{\infty}\mu_{p+k}f_{P+K}(z), \tag{2.22}$$

where

$$\mu_{p+k}\geq 0 \, and \, \sum_{k=-1}^{\infty}\mu_{p+k}=1.$$

Proof .Let $f(z)=\sum_{k=-1}^{\infty}\mu_{p+k}f_{P+K}(z)$, where $\mu_{p+k}\geq 0$ and $\sum_{k=-1}^{\infty}\mu_{p+k}=1$.

Then

$$f(z)=\sum_{k=-1}^{\infty}\mu_{p+k}f_{P+K}(z),$$

$$f(z)=\frac{1}{z^{p}}-\sum_{k=0}^{\infty}M_{P+K}\frac{P(B-A)(P-\alpha)}{(P+K)Q_{k,\lambda}^{n}(a,b;c)M_{k}(\alpha,A,B,P)}z^{p+k}..$$

then

$$\sum_{k=0}^{\infty}\mu_{p+k}\frac{p(B-A)(P-\alpha)}{(P+K)Q_{k,\lambda}^{n}(a,b;c)M_{k}(\alpha,A,B,P)}.\frac{(p+k)Q_{k,\lambda}^{n}(a,b;c)M_{k}(\alpha,A,B,P)}{P(B-A)(p-\alpha)}$$

$$=\sum_{k=0}^{\infty}\mu_{p+k}=1-\mu_{p-1}\leq 1,$$

which shows, that $f(z)\in A_{\lambda,p}^{*n}(a,b,c;\alpha,A,B)$.

Conversely, Let $f(z)\in A_{\lambda,p}^{*n}(a,b,c;\alpha,A,B)$ then

$$a_{k+p}\leq\frac{p(B-A)(P-\alpha)}{(P+K)Q_{k,\lambda}^{n}(a,b;c)M_{k}(\alpha,A,B,P)}.$$

Setting

$$\mu_{p+k}\leq\frac{p(B-A)(P-\alpha)}{(P+K)Q_{k,\lambda}^{n}(a,b;c)M_{k}(\alpha,A,B,P)}a_{k+p} \quad ,$$

and

$$\mu_{p-1}=1-\sum_{k=0}^{\infty}\mu_{p+k}.$$

it follows that $f(z)=\sum_{k=-1}^{\infty}\mu_{p+k}f_{P+K}(z)$. This completes the proof of Theorem.

Theorem 5. *The class* $A_{\lambda,p}^{*n}(a,b,c;\alpha,A,B)$ *is closed under convex linear combinations.*

Proof. Let each of the functions

$$f_{j}(z)=\frac{1}{z^{p}}-\sum_{k=0}^{\infty}a_{k+p,j}z^{p+k} \quad (a_{k+p,j}\geq 0; j=1,2) \tag{2.23}$$

be in the class $A_{\lambda,p}^{*}(a,b,c;\alpha,A,B)$. It sufficient to show that the function $h(z)$ defined by

$$h(z)=(1-t)f_{1}(z)+tf_{2}(z)\in A_{\lambda,p}^{*n}(a,b,c;\alpha,A,B) \; (0\leq t\leq 1), \tag{2.24}$$

is also in the class $A_{\lambda,p}^{*n}(a,b,c;\alpha,A,B)$. since

$$h(z)=\frac{1}{z^{p}}-\sum_{k=0}^{\infty}\left[(1-t)a_{k+p,1}+ta_{k+p,2}\right]z^{k+p} \; (0\leq t\leq 1). \tag{2.25}$$

With the aid of Theorem 1, we have

$$\sum_{k=0}^{\infty}(p+k)Q_{k,\lambda}^{n}(a,b;c)\,M_{k}(\alpha,A,B,P)\left[(1-t)a_{k+p,1}+ta_{p+k,2}\right]$$

$$= (1-t)\sum_{k=0}^{\infty}(p+k)Q_{k,\lambda}^{n}(a,b;c)\ M_k(\alpha,A,B,P)a_{p+k,1}$$

$$+t\sum_{k=0}^{\infty}(p+k)Q_{k,\lambda}^{n}(a,b;c)\ M_k(\alpha,A,B,P)a_{p+k,2}$$

$$\leq (1-t)p(B-A)(P-\alpha)+tp(B-A)(P-\alpha) = p(B-A)(P-\alpha),$$

which shows that $h(z)) \in A_{\lambda,p}^{*n}(a,b,c;\alpha,A,B)$.

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