

Rationalized Haar collocation method for solving singular nonlinear Lane-Emden type equations

A. Pirkhedri^{1*}, H. H. S. Javadi², H. R. Navidi², O. Ghaderi³

¹Department of Computer Engineering, Islamic Azad University, Science and Research Branch, Tehran, Iran

²Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Shahed University, Tehran, Iran

³Department of Computer Engineering, Islamic Azad University, Islamshahr Branch, Tehran, Iran

*Corresponding Author:

A. Pirkhedri

Email: a.pirkhedri@srbiau.ac.ir, alipirkhedri@gmail.com

Abstract: In this study we have proposed the Rationalized Haar (RH) collocation method for the solution of Lane-Emden equations arising in astrophysics as singular initial value problems. In order to test the applicability, accuracy and efficiency of this new method, we considered two examples, for which the comparisons verify our present method. Comparing the results of RH collocation method with the exact solutions clearly indicates that our methods is accurate even when singularity occurs at the boundary.

Keywords: Rationalized Haar; Collocation Method; Lane-Emden Equation; Singularity.

INTRODUCTION

The Lane-Emden equation describes a variety of phenomena in theoretical physics and astrophysics, including aspects of stellar structure [1], the thermal history of a spherical cloud of gas, isothermal gas spheres, and thermionic currents [2] and has been the focus of many studies [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Lane-Emden type equations, first published by Lane [14], and further explored in detail by Emden [15], describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of classical thermodynamics. This equation is a second-order ordinary differential equation with an arbitrary index, known as the polytropic index, that deals with the issue of energy transport, through the transfer of material between different levels of the star. In this study, we consider Lane-Emden differential equations which categorized as singular nonlinear initial value problems of type

$$y''(t) + \frac{\lambda}{t}y'(t) + f(t, y) = g(t), \quad 0 < t < b, \quad \lambda \geq 1 \quad (1)$$

with initial conditions

$$y(0) = \alpha, \quad y'(0) = \beta, \quad (2)$$

where t and y denote the independent and dependent variables and the prime denotes the differentiation with respect to t . The parameters α, β, b and λ is constant, $f(t, y)$ is a nonlinear function of t and y , $g(t)$ is a continuous real valued function in $C[0, b)$. It is well known that an analytic solution of Lane-Emden type equation is always possible [16] in the neighborhood of the singular point $t = 0$ for the above initial conditions. Due to the fact that the Lane-Emden equations are nonlinear and singular, several authors have studied the Adomian and modified decomposition method [3, 4], the homotopy perturbation method [5, 6], the Legendre wavelets [7], the collocation method based on orthogonal Sinc, rational Legendre and Hermite functions method [8, 9, 10], the Pade series method [11], the Bessel collocation method [12] and the variational approach method [13].

In this paper, we propose an efficient numerical method by constructing Rationalized Haar functions and their operational matrices for solving the Lane-Emden type equation (1). The approximate solution obtained by the proposed method shows its superiority on the other existing numerical solution.

The organization of the rest of this letter is as follows: In Section 2, we describe a short introduction to the essentials of RH functions. In Section 3, we summarize the application of RH functions for solving the model equation. In Section 4, the proposed method is applied to some types of Lane-Emden equations, and comparisons are made with the existing analytic solutions that were reported in other published works in the literature. The conclusions and highlights (for review) are described in the sections 5 and 6.

RATIONALIZED HAAR INTERPOLATION

In this section, at first, we introduce RH functions and express some of their basic properties. More, we approximate a function over interval $[0, b)$ using RH functions .

Rationalized Haar Function

The orthogonal set of RH functions is a group of square waves with magnitude of ± 1 in some intervals and zeros elsewhere [17]. The RH functions is defined on the interval $C = [0, b)$ by

$$z_u(t) = \begin{cases} 1, & l_1 \leq t < \frac{l_1}{2}, \\ -1, & \frac{l_1}{2} \leq t < l_0, \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

where

$$l_q = \frac{n-q}{2^m} b, \quad q = 0, \frac{1}{2}, 1. \tag{4}$$

The value of u is defined by two parameters m and n as

$$u = 2^m + n - 1, \quad m = 0, 1, 2, \dots, \quad n = 1, 2, 3, \dots, 2^m. \tag{5}$$

$z_0(t)$ is defined for $m = n = 0$ and is given by

$$z_0(t) = 0, \quad 0 \leq t < b. \tag{6}$$

The set of RH functions is a complete orthogonal set in the Hilbert space $L^2[0, b)$ and orthogonality property is given by

$$\int_0^b z_u(t)z_v(t)dt = \begin{cases} 2^{-m}b, & u = v \\ 0, & u \neq v \end{cases} \tag{7}$$

where

$$v = 2^i + j - 1, \quad i = 0, 1, 2, \dots, \quad j = 1, 2, 3, \dots, 2^i. \tag{8}$$

Function approximation

We can expand any function $y(t) \in L^2[0, b)$ in terms of RH functions as

$$y(t) = \sum_{u=0}^{+\infty} x_u z_u(t), \tag{9}$$

where x_u are given by

$$x_u = \frac{2^m}{b} \int_0^b y(t)z_u(t)dt, \quad u = 1, 2, 3, \dots \tag{10}$$

with $u = 2^m + n - 1, \quad m = 0, 1, 2, 3, \dots, \quad n = 1, 2, 3, \dots, 2^m$ and $u = 0$ for $m = n = 0$.

The series in Eq. (9) contains infinite terms. If, we let $m = 0, 1, 2, \dots, s$, then the infinite series in Eq. (9) is truncated up to its first k terms as

$$y(t) \approx \sum_{u=0}^{k-1} x_u z_u(t) = X^T \zeta(t) \tag{11}$$

where

$$k = 2^{s+1}, \quad s = 0, 1, 2, \dots$$

The RH functions coefficient vector X and RH functions vector $\zeta(t)$ are defined as

$$X = [x_0, \quad x_1, \quad \dots, \quad x_{k-1}]^T, \tag{12}$$

$$\zeta(t) = [\zeta_0(t), \quad \zeta_1(t), \quad \dots, \quad \zeta_{k-1}(t)]^T, \tag{13}$$

where

$$\zeta_u(t) = z_u(t), \quad u = 0, 1, 2, \dots, k - 1. \tag{14}$$

If each waveform is divided into eight intervals, the magnitude of the waveform can be represented as [18]

$$\hat{\chi}_{8 \times 8} = \begin{bmatrix} \zeta_0(t) \\ \zeta_1(t) \\ \vdots \\ \zeta_7(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \tag{15}$$

In Eq. (15) the row denotes the order of the Haar functions. The matrix $\hat{\chi}_{k \times k}$ can be expressed as

$$\hat{\chi}_{k \times k} = \left[\zeta\left(\frac{1}{2k}\right), \zeta\left(\frac{3}{2k}\right), \dots, \zeta\left(\frac{2k-1}{2k}\right) \right] \tag{16}$$

By using Eqs. (11) and (16) we get

$$\left[y\left(\frac{1}{2k}\right), y\left(\frac{3}{2k}\right), \dots, y\left(\frac{2k-1}{2k}\right) \right] = X^T \hat{\chi}_{k \times k}, \tag{17}$$

and we have

$$X^T = \left[y\left(\frac{1}{2k}\right), y\left(\frac{3}{2k}\right), \dots, y\left(\frac{2k-1}{2k}\right) \right] \hat{\chi}_{k \times k}^{-1}, \tag{18}$$

where

$$\hat{\chi}_{k \times k}^{-1} = \binom{1}{k} \text{diag} \left(1, 1, 2, 2, \underbrace{2^2, \dots, 2^2}_{2^2}, \underbrace{2^3, \dots, 2^3}_{2^3}, \dots, \underbrace{\frac{k}{2}, \dots, \frac{k}{2}}_{\frac{k}{2}} \right) \tag{19}$$

Operational matrix of integration

The integration of the $\zeta(t)$ defined in Eq. (13) is given by

$$\int_0^t \zeta(\ell) d\ell \approx J\zeta(t), \tag{20}$$

where $J = J_{k \times k}$ is the $k \times k$ operational matrix for integration and is given in [19] as

$$J_{k \times k} = \frac{b}{2k} \begin{bmatrix} 2kJ_{\left(\frac{k}{2}\right) \times \left(\frac{k}{2}\right)} & -\hat{\chi}_{\left(\frac{k}{2}\right) \times \left(\frac{k}{2}\right)} \\ \hat{\chi}_{\left(\frac{k}{2}\right) \times \left(\frac{k}{2}\right)}^{-1} & 0 \end{bmatrix}, \tag{21}$$

where $\hat{\chi}_1 = [1], J_1 = \left[\frac{1}{2}\right]$.

For example, the matrices $J_{8 \times 8}$ can be written as follows:

$$J_{8 \times 8} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{4} & \frac{-1}{8} & \frac{-1}{8} & \frac{-1}{16} & \frac{-1}{16} & \frac{-1}{16} & \frac{-1}{16} \\ \frac{1}{4} & 0 & \frac{-1}{8} & \frac{1}{8} & \frac{-1}{16} & \frac{-1}{16} & \frac{1}{16} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{8} & 0 & 0 & \frac{-1}{16} & \frac{1}{16} & 0 & 0 \\ \frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 & \frac{-1}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 & \frac{-1}{16} & \frac{1}{16} \\ \frac{1}{64} & \frac{1}{64} & \frac{1}{32} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{64} & \frac{1}{64} & \frac{-1}{32} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{64} & \frac{1}{64} & \frac{-1}{32} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{64} & \frac{-1}{64} & 0 & \frac{1}{32} & 0 & 0 & 0 & 0 \\ \frac{1}{64} & \frac{1}{64} & 0 & \frac{1}{32} & 0 & 0 & 0 & 0 \\ \frac{1}{64} & \frac{-1}{64} & 0 & \frac{-1}{32} & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{22}$$

SOLVING LANE-EMDEN EQUATION

Consider the following Lane-Emden equation

$$y''(t) + \frac{\lambda}{t} y'(t) + f(t, y(t)) = g(t), \quad 0 < t < b, \quad \lambda \geq 1 \tag{23}$$

$$y(0) = \alpha, \quad y'(0) = \beta \tag{24}$$

To solve Eq. (23) with boundary conditions in Eq. (24) we let

$$y''(t) = \sum_{u=0}^{k-1} x_u z_u(t) = X^T \zeta(t) \tag{25}$$

where

$$u = 2^m + n - 1, \quad m = 0, 1, 2, \dots, \quad n = 1, 2, 3, \dots, 2^m. \tag{26}$$

Using Eqs. (20) and (25) we get

$$y'(t) = X^T J \zeta(t) + \beta, \tag{27}$$

and

$$y(t) = X^T J \int_0^t \zeta(\ell) d\ell + \beta t + \alpha = X^T J^2 \zeta(t) + \beta t + \alpha \tag{28}$$

Using Eqs. (23)-(28) we get

$$X^T \zeta(t) + \frac{\lambda}{t} (X^T J \zeta(t) + \beta) + f(t, X^T J^2 \zeta(t) + \beta t + \alpha) = g(t), \tag{29}$$

and therefore

$$X^T \left[\zeta(t) + J \frac{\lambda}{t} \zeta(t) + \frac{\lambda}{t} \beta \right] + f(t, X^T J^2 \zeta(t) + \beta t + \alpha) = g(t). \tag{30}$$

The residual $Res_k(t)$ for Eq. (23) can be written as:

$$Res_k(t) \approx X^T \left[\zeta(t) + J \frac{\lambda}{t} \zeta(t) + \frac{\lambda}{t} \beta \right] + f(t, X^T J^2 \zeta(t) + \beta t + \alpha) - g(t). \tag{31}$$

The equations for obtaining the coefficients x_u arises from equalizing $Res_k(t)$ to zero at k RH collocation points defined by:

$$t_i = \frac{2i-1}{2k} b, \quad i = 1, 2, \dots, k. \tag{32}$$

By substitution collocation points $t_i, i = 1, 2, \dots, k$ in $Res_k(t)$ and equalizing to zero we have

$$Res_k(t_i) = 0, \quad i = 1, 2, \dots, k. \tag{33}$$

Eq. (33) gives k nonlinear algebraic equations which can be solved for the unknown coefficients $x_u, u = 0, 1, \dots, k - 1$ by using the wellknown Newton's method. Consequently, $y(t)$ given in Eq. (23) can be calculated.

ILLUSTRATIVE EXAMPLES

In this section, we applied the method presented in this paper for solving Eq. (1) and showed the efficiency of the method with the numerical results of two examples.

Example 1

Consider the Lane-Emden equation given in [20] by

$$y''(t) + \frac{8}{t}y'(t) + ty(t) = t^5 - t^4 + 44t^2 - 30t, \quad 0 < t < 1, \tag{34}$$

$$y(0) = 0, \quad y'(0) = 0,$$

with the analytical solution $y(t) = t^4 - t^3$.

This type of equation has been solved with Hermite functions collocation [10] homotopy analysis [21], and linearization method [22] respectively.

Table 1 shows the comparison of the $y(t)$ between numerical solutions obtained by the method proposed in this letter for $k = 8, 16, 32$ and their absolute errors with respect to the exact solution. The approximate solution by the presented method with $k = 32$ is shown in Fig. 1 to make it easier to compare with the analytic solution. The logarithmic graphs of absolute coefficients $|x_u|$ of RH functions for $k = 16, 32$ are shown in Figs. 2 and 3.

Example 2

Consider the Lane-Emden equation given in [20] by

$$y''(t) + \frac{2}{t}y'(t) - 2(2t^2 + 3)y(t) = 0, \quad 0 < t \leq 1, \tag{35}$$

$$y(0) = 1, \quad y'(0) = 0,$$

with the analytical solution $y(t) = e^{t^2}$.

By applying the presented method for different values of k , we obtained the approximate solutions in Table 2. The resulting graph of Eq.(35) with present method for $k = 32$ in comparison to the exact solution is shown in Fig. 4. The logarithmic graph of the absolute coefficients of RH functions for $k = 32$ is shown in Fig. 5.

Fig. 6 explicate the rapid convergence of the maximum errors for RH collocation method with the increase in the orders of approximation for $k = 4, 8, 16$ and 32.

Table 1: Comparison of absolute errors of Example 1 for different values of k

| t | Exact | Absolute error= $ y_{exact} - y_{RH} $ | | |
|-----|---------|--|------------|-----------|
| | | k=8 | k=16 | k=32 |
| 0.1 | -0.0009 | 0.0004610 | 0.00010955 | 0.0000200 |
| 0.2 | -0.0064 | 0.0007937 | 0.00013490 | 0.0000332 |
| 0.3 | -0.0189 | 0.0006500 | 0.00016100 | 0.0000380 |
| 0.4 | -0.0384 | 0.0005230 | 0.00014900 | 0.0000350 |
| 0.5 | -0.0625 | 0.0004380 | 0.00011200 | 0.0000220 |
| 0.6 | -0.0864 | 0.0003060 | 0.00005400 | 0.0000010 |
| 0.7 | -0.1029 | 0.0001100 | 0.00007000 | 0.0000300 |
| 0.8 | -0.1024 | 0.0008200 | 0.00015000 | 0.0000600 |
| 0.9 | -0.0729 | 0.0016250 | 0.00035100 | 0.0000990 |
| 1.0 | 0.000 | 0.0021553 | 0.00052301 | 0.0001533 |

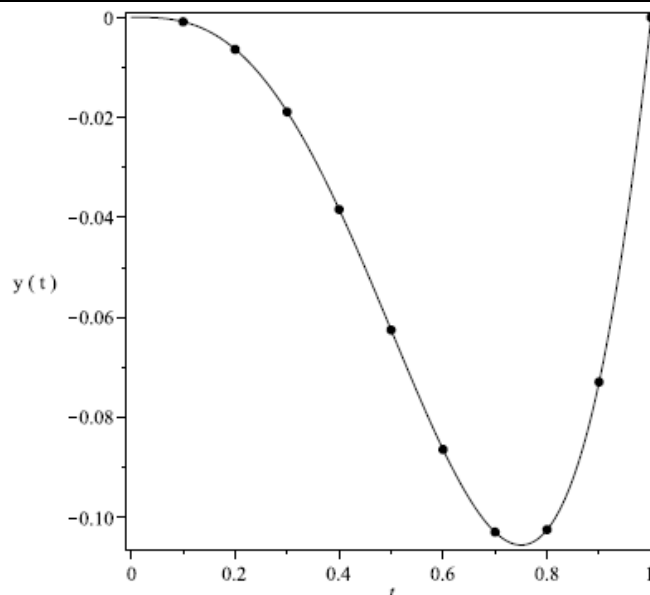


Fig-1: Graph of equation of Example 1 obtained by present method (solid line) in comparison with exact solution (circles-solid) for $k = 32$

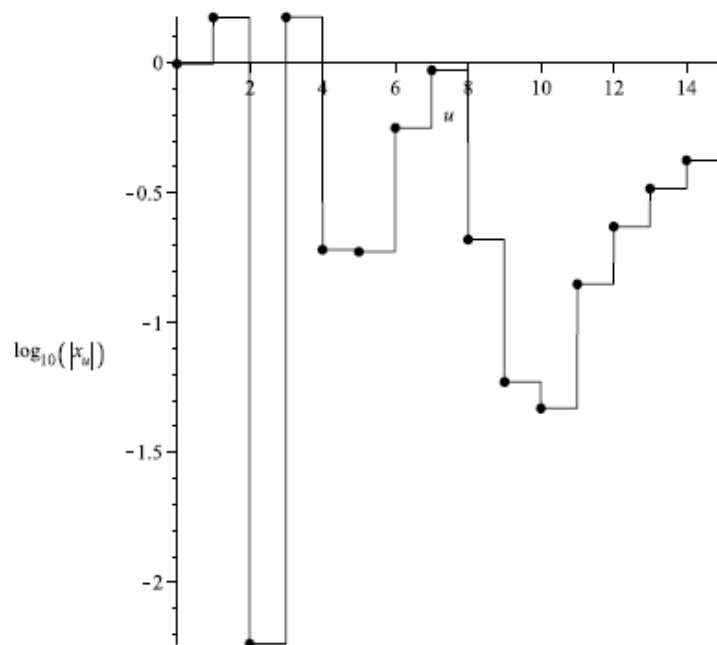


Fig-2: Logarithmic graph of absolute coefficients $|x_u|$ of RH functions of Example 1 for $k = 16$

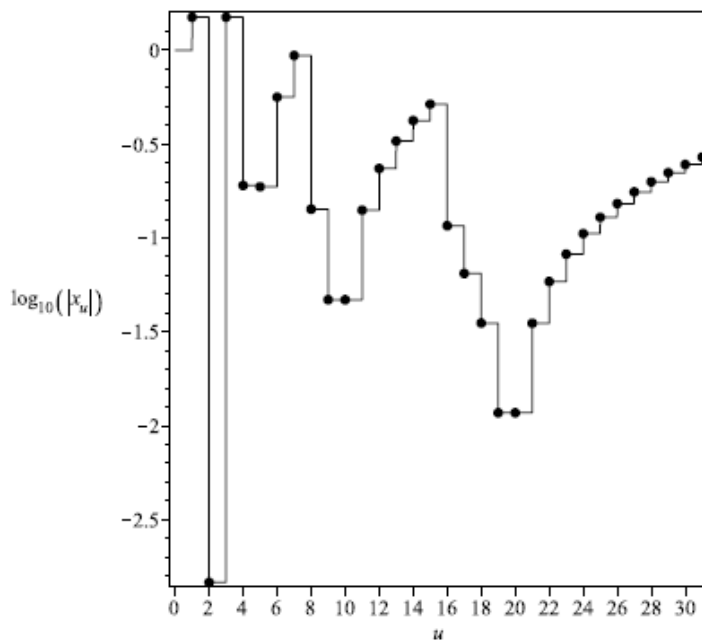


Fig-3: Logarithmic graph of absolute coefficients $|x_u|$ of RH functions of Example 1 for $k = 32$

Table-2: Numerical approximations of Example 2 for different values of k

| t | Exact | \mathcal{Y}_{RH} | | |
|-----|-----------|--------------------|-----------|-----------|
| | | k=8 | k=16 | k=32 |
| 0.1 | 1.0100502 | 1.0100655 | 1.0100495 | 1.0100497 |
| 0.2 | 1.0408108 | 1.0408035 | 1.0407936 | 1.0408073 |
| 0.3 | 1.0941743 | 1.0940582 | 1.0941545 | 1.0941677 |
| 0.4 | 1.1735109 | 1.1732329 | 1.1734575 | 1.1735000 |
| 0.5 | 1.2840254 | 1.2837228 | 1.2839504 | 1.2840067 |
| 0.6 | 1.4333294 | 1.4331505 | 1.4332458 | 1.4333013 |
| 0.7 | 1.6323162 | 1.6320688 | 1.6321618 | 1.6322840 |
| 0.8 | 1.8964809 | 1.8959220 | 1.8964077 | 1.8964529 |
| 0.9 | 2.2479080 | 2.2470309 | 2.2477875 | 2.2478930 |
| 1.0 | 2.7182818 | 2.7182501 | 2.7182809 | 2.7182821 |

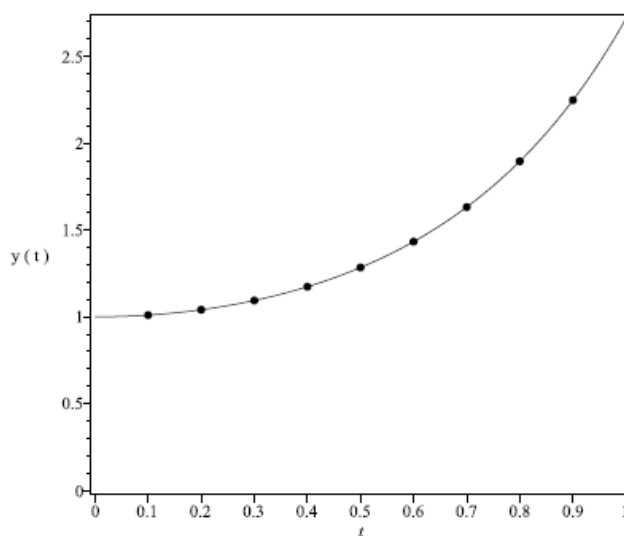


Fig-4: Graph of equation of Example 2 obtained by present method (solid line) in comparison with exact solution (circles-solid) for $k = 32$

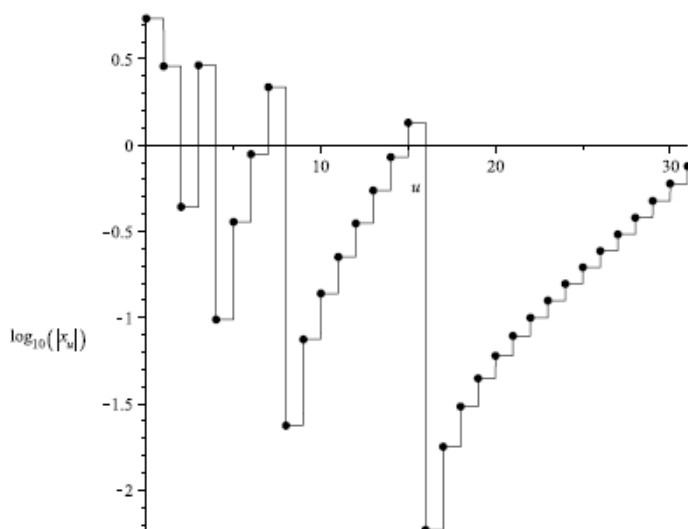


Fig-5: Logarithmic graph of absolute coefficients $|x_u|$ of RH functions of Example 2 for $k = 32$

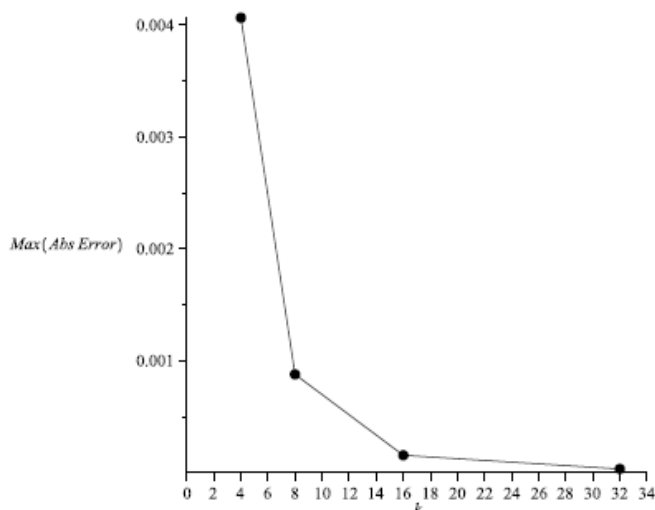


Fig-6: Error convergence with the different values of k for Examples 2

CONCLUSIONS

In this study, we used the RH collocation method to obtain numerical solutions of a nonlinear ODE's related to the astrophysics problems. The properties of the RH functions were utilized to reduce the computation of Lane-Emden equations to the solution of algebraic equations with unknown coefficients. The effectiveness of the method was investigated by comparing the results obtained with the exact solutions. we also showed our method is easy to implement and yields very accurate results.

HIGHLIGHTS

- The RH functions are notable for their rapid convergence for the function expansion.
- The proposed method can handle the singularity that occurs at the boundary.
- For the first time the collocation method based on RH functions has been applied.
- The proposed method is utilized to reduce the problem to the solution of algebraic equations.
- Tables and figures show that the errors decrease as the values of k increase.

REFERENCES

1. Chandrasekhar S; Introduction to the Study of Stellar Structure, Dover, New York, 1967.
2. Richardson OU; The Emission of Electricity from Hot Bodies, Longman, Green and Co., London, New York, 1921.
3. Wazwaz AM; A new algorithm for solving differential equations of Lane–Emden type. Applied Mathematics and Computation, 2001;118(2):287-310.
4. Wazwaz AM;The modified decomposition method for analytic treatment of differential equations. Applied Mathematics and Computation, 2006;173(1):165-176.

5. Hojjati G, Parand K; An efficient computational algorithm for solving the nonlinear Lane-Emden type equations. *International Journal of Mathematics and Computation*, 2011; 4(7):182-187.
6. Momoniat E, Harley C; An implicit series solution for a boundary value problem modelling a thermal explosion. *Mathematical and Computer Modelling*, 2011; 53(1):249-260.
7. Yousefi SA; Legendre wavelets method for solving differential equations of Lane-Emden type. *Applied Mathematics and Computation*, 2006;181(2):1417-1422.
8. Parand K, Pirkhedri A; Sinc-collocation method for solving astrophysics equations. *New Astronomy*, 2010;15(6): 533-537.
9. Parand K, Shahini M, Dehghan M; Rational Legendre pseudospectral approach for solving nonlinear differential equations of Lane-Emden type. *Journal of Computational Physics*, 2009; 228(23):8830-8840.
10. Parand K, Dehghan M, Rezaei AR, Ghaderi SM; An approximation algorithm for the solution of the nonlinear Lane-Emden type equations arising in astrophysics using Hermite functions collocation method. *Computer Physics Communications*, 2010; 181(6):1096-1108.
11. Vanani SK, Aminataei A; On the numerical solution of differential equations of Lane-Emden type. *Comput. Math. Appl.*, 2010;59(8): 2815 - 2820.
12. Yüzbaşı Ş, Sezer M; A collocation approach to solve a class of Lane-Emden type equations. *J. Adv. Res. Appl. Math*, 2011;3(2):58-73.
13. He JH; Variational approach to the Lane-Emden equation. *Applied Mathematics and Computation*, 2003;143(2):539-541.
14. Pandey RK, Bhardwaj A, Kumar N; Solution of Lane-Emden type equations using Chebyshev wavelet operational matrix. *Journal of Advanced Research in Scientific Computing*, 2012; 4(1).
15. Emden R; *Gaskugeln* Teubner, Leipzig and Berlin, 1907.
16. Davis HT; *Introduction to Nonlinear Differential and Integral Equations*, Dover, New York, 1962.
17. Hariharan G; Wavelet method for a class of fractional Klein-Gordon equations. *Journal of Computational and Nonlinear Dynamics*, 2013; 8(2): 021008.
18. Ohkita M, Kobayashi Y; Piecewise-linear approximation of solution of linear differential equations by Walsh functions. *Mathematics and Computers in Simulation*, 1990; 32(3):297-308.
19. Razzaghi M, Ordokhani Y; Solution of nonlinear Volterra-Hammerstein integral equations via rationalized Haar functions. *Mathematical Problems in Engineering*,2001;7(2):205-219.
20. Wazwaz AM; A new method for solving singular initial value problems in the second-order ordinary differential equations. *Applied Mathematics and Computation*, 2002; 128(1):45-57.
21. Ramos JJ; Linearization techniques for singular initial-value problems of ordinary differential equations. *Applied Mathematics and Computation*, 2005;161(2):525-542.
22. Bataineh AS, Noorani MSM, Hashim I; Homotopy analysis method for singular IVPs of Emden-Fowler type. *Communications in Nonlinear Science and Numerical Simulation*, 2009; 14(4):1121-1131.