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# Diamond Osculating Planes of Curves on Time Scales

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**Abstract:** In this paper, we present normal, binormal, and osculating plane of diamond regular curves on time scales. We also study their equations analytically.
**Keywords:** Time Scale Calculus, Differential Theory, Local Analysis, Symmetric Differential.

## INTRODUCTION

A time scale can be defined as non-empty closed subsets of reals [2]. It's the theory that has the benefits of unification of continuous and discrete calculus. The concept of dynamic equations has motivated a huge size of research work in recent years [3,5,9]. Geometric interpretation of the theory of timescales is extensively studied afterwards the introduction of partial derivatives on time scales [4,7,10].

In [6], authors presented the symmetric derivative on time scales and its relation to forward and backward derivatives. This study aims the differentiability of the functions where their derivatives vanish. This kind of calculus also come up with more precisely and well defined tangent line definition [10].

In this study, we purpose the idea of osculating planes of a regular curve on time scales. For this purpose, we first introduce the concept of vector valued functions on time scales in Section 2. We also analyze this kind of functions by symmetric, or as known as diamond, derivatives of their real valued coordinate functions. We also present an equation for a diamond tangent line, by using partial symmetric derivatives on time scales. More on partial derivatives can be found in [10]. In Section 3, we use diamond derivatives to analytically define osculating planes besides the normal and binormal planes.

## VECTOR VALUED FUNCTIONS

Let $\mathrm{T}_i$ denote an arbitrary time scale, for $i = 1, 2, \ldots, n$. The natural tensor product of this time scales lead us an $n$-dimensional time scale

$$\Lambda^n = \mathrm{T}_1 \times \mathrm{T}_2 \times \ldots \times \mathrm{T}_n = \left\{ \left( t_1, t_2, \ldots, t_n \right) : t_i \in \mathrm{T}_i, \forall i \in \{1, 2, \ldots, n\} \right\}.$$

To analyze vector valued functions via the symmetric derivative on time scales, we may first define vector valued function as a mapping from a time scale to $n$-dimensional real space, i.e.,

$$\phi : \mathrm{T} \to R^n, \quad t \mapsto \phi(t) = \{\phi_1(t), \phi_2(t), \ldots \phi_n(t)\},$$

where $\phi_1(t), \phi_2(t), \ldots \phi_n(t)$ are real-valued coordinate functions. Limit of this vector valued function can be defined as in [4, Definition 2.1]. The symmetric derivative of such function can also be defined as same fashion.

**Definition 2.1:** $\Diamond$-derivative of a vector valued function can be defined by the $\Diamond$-derivative of each coordinate functions, i.e.,

$$\phi^{\Diamond}(t) = \left\{ \phi_1^{\Diamond}(t), \phi_2^{\Diamond}(t), \ldots, \phi_n^{\Diamond}(t) \right\}.$$

More precisely, if the limit value

$$\lim_{s \to t} \frac{\phi(\sigma(\mathrm{t})) - \phi(s) + \phi(2\mathrm{t} - s) - \phi(\rho(s))}{\sigma(\mathrm{t}) + 2t - 2s - \rho(s)}$$

exists as a finite number, we may call $\phi$ as symmetric differentiable vector valued function at $t \in \mathrm{T}_{\kappa}^{\kappa}$

**Proposition 2.1:** Let $\phi(t)$ $\varphi(t)$ and be two vector valued functions for $t \in \mathrm{T}_{\kappa}^{\kappa}$, $\cdot$ and $\times$ denote Euclidean inner and cross product, respectively. Then,

i. $\left(\phi(t)\cdot\varphi(t)\right)^{\Diamond} = \phi^{\Diamond}(t)\cdot\varphi(\sigma(t)) + \phi(\rho(t))\cdot\varphi^{\Diamond}(t)$

ii. $\left(\phi(t)\times\varphi(t)\right)^{\Diamond} = \phi^{\Diamond}(t)\times\varphi(\sigma(t)) + \phi(\rho(t))\times\varphi^{\Diamond}(t)$

The $\Diamond$-differentiation of the inner products and vector products of vector-valued functions can be computed by the consecutive $\Diamond$-differentiation of the cofactors.

**Definition 2.2:** Assume that $k$ times $\Diamond$-derivative of the vector-valued function $\phi(t)$ exists and are time scale-continuous, then we can write Taylor's expansions for the components $\{\phi_1(t), \phi_2(t), \ldots, \phi_n(t)\}$ as

$$\phi_1(t) = h_0(t,t_0)\phi_1(t_0) + h_1(t,t_0)\phi_1^{\Diamond}(t_0) + h_2(t,t_0)\phi_1^{\Diamond^2}(t_0) + \cdots + h_k(t,t_0)\phi_1^{\Diamond^k}(t_0) + o_1\left(\phi_1^{\Diamond^{k+1}}\right)$$
$$\vdots$$
$$\phi_n(t) = h_0(t,t_0)\phi_n(t_0) + h_1(t,t_0)\phi_n^{\Diamond}(t_0) + h_2(t,t_0)\phi_n^{\Diamond^2}(t_0) + \cdots + h_k(t,t_0)\phi_n^{\Diamond^k}(t_0) + o_n\left(\phi_n^{\Diamond^{k+1}}\right),$$

where, $t, t_0 \in \mathrm{T}_{\kappa}^{\kappa}$, $h_0(t,s) \equiv 1$, $h_{k+1}(t,t_0) = \int_{t_0}^{t} h_k(\tau,t_0)\Diamond\tau$ .

For more diamond integration on time scales see [10].

This system of $n$ equations can be written as

$$\phi(t) = h_0(t,t_0)\phi(t_0) + h_1(t,t_0)\phi^{\Diamond}(t_0) + h_2(t,t_0)\phi^{\Diamond^2}(t_0) + \cdots + h_k(t,t_0)\phi^{\Diamond^k}(t_0) + o\left(\phi^{\Diamond^{k+1}}(t_0)\right),$$

where $o\left(\phi^{\Diamond^{k+1}}(t_0)\right)$ denotes a vector whose length is infinitesimal.

**Definition 2.3:** Let $\mathrm{T}$ be a time scale. A diamond regular curve $\gamma$ is defined as a mapping

$$\gamma : \mathrm{T} \to R^3, \quad t \mapsto \gamma(t) = \left(\gamma_1(t), \gamma_2(t), \gamma_3(t)\right)$$

for $t \in [a,b] \subseteq \mathrm{T}_{\kappa}^{\kappa}$ with $\left|\gamma_1^{\Diamond}\right|^2 + \left|\gamma_2^{\Diamond}\right|^2 + \left|\gamma_3^{\Diamond}\right|^2 \neq 0.$

**Definition 2.4:** Let $\gamma : \mathrm{T} \to R^3$ be a real valued diamond regular curve and $t_0 \in \mathrm{T}_{\kappa}^{\kappa}$. The line with the slope $\gamma^{\Diamond}(t_0)$ passing at the point $\gamma(t_0)$ is called Diamond tangent line of $\gamma$ . See [10].

Now, let three coordinate functions for a diamond regular curve $\gamma_1 : \mathrm{T} \to R$ , $\gamma_2 : \mathrm{T} \to R$ , and $\gamma_3 : \mathrm{T} \to R$ be given. Let us set $\gamma_1(\mathrm{T}) := \mathrm{T}_1$ , $\gamma_2(\mathrm{T}) := \mathrm{T}_2$ , $\gamma_3(\mathrm{T}) := \mathrm{T}_3$ . It is natural to assume that $\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3$ are time scales. With these assumptions, let us define two closed form functions:

$$\phi, \varphi : \mathrm{T}_1 \times \mathrm{T}_2 \times \mathrm{T}_3 \to R$$
$$\phi(\gamma_1, \gamma_2, \gamma_3) = 0, \quad \varphi(\gamma_1, \gamma_2, \gamma_3) = 0$$

which lead us a space curve. If we substitute the position vectors of the considered curve, then we obtain two equalities:

$$\phi(\gamma_1(t), \gamma_2(t), \gamma_3(t)) = 0$$
$$\varphi(\gamma_1(t), \gamma_2(t), \gamma_3(t)) = 0.$$

If the functions $f$ and $j$ are diamond differentiable, then

$$\begin{aligned} &\frac{\partial\phi}{\Diamond_1\gamma_1}\gamma_1^{\Diamond}+\frac{\partial\phi}{\Diamond_2\gamma_2}\gamma_2^{\Diamond}+\frac{\partial\phi}{\Diamond_3\gamma_3}\gamma_3^{\Diamond}=0 \\ &\frac{\partial\varphi}{\Diamond_1\gamma_1}\gamma_1^{\Diamond}+\frac{\partial\varphi}{\Diamond_2\gamma_2}\gamma_2^{\Diamond}+\frac{\partial\varphi}{\Diamond_3\gamma_3}\gamma_3^{\Diamond}=0, \end{aligned} \quad (1)$$

where $\Diamond_1$, $\Diamond_2$ ,and $\Diamond_3$ are the partial symmetric derivative operator for $\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3$, respectively [7,10]. The components $\{\gamma_1^{\Diamond}, \gamma_2^{\Diamond}, \gamma_3^{\Diamond}\}$ of diamond tangent vector satisfy the system of consisting equations (1).

For a planar curve $\gamma$, given by the equations $\gamma(\gamma_1,\gamma_2)=0,\ \gamma_3=0$ satisfying the condition $(\partial\phi/\Diamond_1\gamma_1)^2+(\partial\phi/\Diamond_2\gamma_2)^2\neq 0$; then the components of the tangent vector $\{\gamma_1^{\Diamond}, \gamma_2^{\Diamond}\}$ satisfies the given equation:

$$\frac{\partial\phi}{\Diamond_1\gamma_1}\gamma_1^{\Diamond}+\frac{\partial\phi}{\Diamond_2\gamma_2}\gamma_2^{\Diamond}=0.$$

Therefore, $\{\gamma_1^{\Diamond},\gamma_2^{\Diamond}\}=\mu\{-\partial\phi/\Diamond_2\gamma_2, \partial\phi/\Diamond_1\gamma_1\}$, and the equation of the diamond tangent is

$$\frac{x-\gamma_1}{-\partial\phi/\Diamond_2\gamma_2}=\frac{y-\gamma_2}{\partial\phi/\Diamond_1\gamma_1},$$

where $x$ and $y$ are standard Euclidean coordinate functions.

**OSCULATING PLANES**

**Definition 3.1:** Let $\gamma$ be a regular and diamond differentiable space curve. The plane passing through the point $P_0\in\gamma$ and orthogonal to the vector tangent to $\gamma$ at $P_0$ is called the plane normal to $\gamma$ at $P_0$. The plane with the normal direction $\gamma^{\Diamond}(P_0)$ and orthogonal the normal plane of $\gamma$ at $P_0$ is called the binormal plane.

Let $\hat{\gamma}$ denote the position vector of normal plane. Since this plane is orthogonal to the vector $\gamma^{\Diamond}$ and contains the point with the position vector $\hat{\gamma}-\gamma(t_0)$, the equation of the normal plane is

$$(\hat{\gamma}-\gamma(t_0))\cdot\gamma^{\Diamond}(t_0)=0.$$

With the similar fashion one may obtain the equation of binormal plane as

$$\gamma_1^{\Diamond}(t_0)x+\gamma_2^{\Diamond}(t_0)y+\gamma_3^{\Diamond}(t_0)z=0,$$

where $\{x,y,z\}$ are the standard Euclidean coordinate functions.

**Theorem 3.1:** Let $\gamma$ be a regular and represented as $\gamma=\gamma(t)$. Assume that the vectors $\gamma^{\Diamond}$ and $\gamma^{\Diamond^2}$ are not collinear at $\gamma(t_0)$. Then there exists osculating plane of $\gamma$ at $\gamma(t_0)$ and is spanned by the vectors $\gamma^{\Diamond}$ and $\gamma^{\Diamond^2}$.

***Proof.*** If $t_0$ is a dense point, then diamond derivative turns to be usual derivative and proof can be completed as in classical differential geometry, see [1].

Let $t_0$ be a scattered point. Then, the position vectors of $\overrightarrow{P_0Q_1}$ and $\overrightarrow{P_0Q_2}$ are $a_1=\gamma(t_0+\tau_1)$ and $a_2=\gamma(t_0+\tau_2)$, respectively. That is, if these vectors are linearly independent, then they span such a plane $\Omega$. This plane is also spanned by the vectors $\upsilon^{(1)}/a_1$ and $\upsilon^{(2)}/a_2$. One may also conclude the relation of

$$\upsilon^{(1)},\ \omega=\frac{2(\upsilon^{(2)}-\upsilon^{(1)})}{\tau_2-\tau_1}$$

span the $\Omega$ . If we take Taylor's expansion in the account; i.e.,

$$\gamma(t) = h_0(t,t_0)\gamma(t_0) + h_1(t,t_0)\gamma^{\Diamond}(t_0) + h_2(t,t_0)\gamma^{\Diamond^2}(t_0) + o(E),$$

we obtain

$$\upsilon^{(1)} = \gamma^{\Diamond}(t_0) + \frac{\tau_1}{2}\gamma^{\Diamond^2}(t_0) + \mathrm{o}(\tau_1)$$

$$\omega = \gamma^{\Diamond^2}(t_0) + \mathrm{o}(1).$$

Consequently, if $\tau_1$ and $\tau_2$ approach to zero, then $\upsilon^{(1)} \to \gamma^{\Diamond}(t_0)$ and $\omega \to \gamma^{\Diamond^2}(t_0)$ . □

**Corollary 3.1:** The osculating plane of a planar curve coincides with the plane containing this curve.

By this idea, it is possible to obtain the equation of the osculating plane of a regular curve. Let $\breve{\gamma}$ denote the position vector of the osculating plane. Since $\gamma^{\Diamond}$ and $\gamma^{\Diamond^2}$ span the osculating plane, the vector $\gamma^{\Diamond} \times \gamma^{\Diamond^2}$ is orthogonal to this plane. Therefore,

$$\left(\breve{\gamma}(t_0) - \gamma(t_0)\right) \cdot \left(\gamma^{\Diamond} \times \gamma^{\Diamond^2}\right) = 0 .$$

By the standard Euclidean coordinate functions $\{x, y, z\}$ , this equation yields

$$\det \begin{pmatrix} x - \gamma(t_0) & \gamma_1^{\Diamond} & \gamma_1^{\Diamond^2} \\ y - \gamma(t_0) & \gamma_2^{\Diamond} & \gamma_2^{\Diamond^2} \\ z - \gamma(t_0) & \gamma_3^{\Diamond} & \gamma_3^{\Diamond^2} \end{pmatrix} = 0.$$

## CONCLUSIONS

In this study, we present a new technique to define analytic equations of regular curves on time scales. For this purpose, we use the symmetric derivative on time scales that is introduced in [6]. Since the diamond differentiability does not yield restrictions as completely differentiability [10], this kind of calculus help us to obtain equations precisely. The main disadvantage of restrictions can be seen in [4]. It's possible to apply this method to obtain some other characteristics of regular curves on time scales.

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