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Application of Halanay Inequality to the Stability of the Disease Free Equilibrium of a Delayed Malaria Transmission Model

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Abstract: By the applications of Halanay type inequality and the theory of nonsingular M-matrix, the global asymptotical stability of the disease free equilibrium of a delayed malaria transmission model is obtained when the basic reproduction number of the model is less than 1.

Keywords: Halanay inequality, Global asymptotical stability, Malaria transmission, Time delay, Nonsingular M-matrix.

INTRODUCTION

Stability properties, either local or global, of the equilibrium of infectious disease models are key important in the quantitative analysis for the disease transmission [5, 8, 9, 11, 12, 15]. Those mathematical models are now usually involved with two or more delays [8, 9, 11], which greatly increase the difficulty to analyze the stability, especially the global asymptotical stability, of the equilibrium of the models. The local asymptotical stability of the equilibrium can be obtained by analysis of the roots of the characteristic equation of the linearized model at the corresponding equilibrium [3, 6], and the global stability can be obtained by constructing suitable Liapunov functions together with Razumikhin type theorems [4, 6]. In the literature for stability analysis for the equilibrium of infectious disease models with two or more delays, the local asymptotical stability is often achieved while the global asymptotical stability remained unsolved due to the fact that it is not easy to construct a suitable Liapunov function for the models with several delays [8, 9, 11].

It is well known that the Halanay type inequalities can be used to obtain the global exponential stability of the equilibrium of a delayed mathematical model [1, 2, 10, 14], and it is widely used in the stability analysis of the equilibrium of models established in neural networks [1, 2, 14]. Also, it is common knowledge that the equilibrium is globally asymptotically stable if it is globally exponentially stable [4]. Motivated by the above, in this paper we shall obtain the global asymptotical stability of the disease free equilibrium of a malaria transmission model with two delays by the application of Halanay type inequality.

For compartmental infectious disease models, it is a common theory that the disease free equilibrium is globally asymptotically stable when the basic reproduction number is less than 1 and the positive equilibrium exists, which is globally asymptotically stable when the basic reproduction number is larger than 1 [5]. By the application of Halanay type inequality we can obtain that the disease free equilibrium of the delayed malaria transmission model in this paper is globally asymptotically stable when the basic reproduction number is less than 1, which is well consistent with the common theory of compartmental infectious disease models.

PRELIMINARIES

The following model was established in [9] to reflect the transmission of malaria between human and mosquito population

$$\begin{cases} \dfrac{dx}{dt} = -\gamma x + abm\exp(-\gamma\tau_1)[1-x(t-\tau_1)]y(t-\tau_1), \\ \dfrac{dy}{dt} = -\mu y + ac\exp(-\mu\tau_2)[1-y(t-\tau_2)]x(t-\tau_2). \end{cases} \tag{1}$$

We introduce the notations in (1) briefly, one can refer to [9] for more details about (1). In (1), $x(t)$ $(y(t))$ is the ratio of the number of the infected human (mosquito) to the total number of human (mosquito) population at time t (unit: days),

respectively and it is supposed that the total number of human (mosquito) population is constant H (M), $m = M/H$, a is the average bites of a mosquito per human per day, b is the rate of a susceptible human becoming infectious after the bite by an infected mosquito, c is the rate of a susceptible mosquito becoming infectious after its bite of an infected human, γ is the recovery rate of human from the infection, μ is the natural death rate of the mosquito, and τ_1, τ_2 is the incubation time of the parasites in the human, mosquito body, respectively. The parameters $a, b, c, m, \gamma, \mu, \tau_1, \tau_2$ are all positive.

The basic reproduction number R_0 of (1) is defined as

$$R_0 = \frac{a^2 bcm \exp(-\gamma\tau_1)\exp(-\mu\tau_2)}{\gamma\mu}. \tag{2}$$

By direct computation, one has

Lemma 2.1. (i) There exists only disease free equilibrium $E_0 = (0, 0)$ of (1) when $R_0 \le 1$; (ii) when $R_0 > 1$, there exists the disease free equilibrium E_0 and a unique positive equilibrium $E^* = (x^*, y^*)$, where

$$x^* = \frac{R_0 - 1}{R_0 + ac\exp(-\mu\tau_2)/\mu}, \quad y^* = \frac{R_0 - 1}{R_0 + abm\exp(-\gamma\tau_1)/\gamma}. \tag{3}$$

The initial conditions of (1) are given as follows

$$x(s) = \phi(s),\ y(s) = \psi(s),\ s \in [-\tau, 0],\ \tau = \max\{\tau_1, \tau_2\}, \tag{4}$$

where $\phi(s), \psi(s)$ is continuous in $[-\tau, 0]$ and

$$0 < \phi(s) < 1,\ 0 < \psi(s) < 1. \tag{5}$$

It is proved in [9] that E_0 is locally asymptotically stable if $R_0 < 1$, but the global asymptotical stability of E_0 is not analyzed there.

Lemma 2.2. If

$$abm\exp(-\gamma\tau_1)/\gamma < 1,\ ac\exp(-\mu\tau_2)/\mu < 1, \tag{6}$$

then the solutions of (1) with (5) satisfy

$$0 < x(t) < 1,\ 0 < y(t) < 1,\ t \ge 0. \tag{7}$$

Proof. We first prove that $0 < x(t) < 1$ for $t \in [0, \tau_1]$. Suppose the contrary, there exists a $\bar{t} \in [0, \tau_1]$ such that $x(\bar{t}) = 0$ and $x(t) > 0$ for $t < \bar{t}$. Thus, we have $dx(\bar{t})/dt \le 0$. But

$$\begin{aligned}\frac{dx(\bar{t})}{dt} &= -\gamma x(\bar{t}) + abm\exp(-\gamma\tau_1)(1 - x(\bar{t} - \tau_1))y(\bar{t} - \tau_1)\\ &= -\gamma\cdot 0 + abm\exp(-\gamma\tau_1)(1 - \phi(\bar{t} - \tau_1))\psi(\bar{t} - \tau_1)\\ &> 0\end{aligned}$$

since (5). This is a contradiction. Hence, $x(t) > 0$ for $t \in [0, \tau_1]$.

Next, suppose that there exists a $\hat{t} \in [0, \tau_1]$ such that $x(\hat{t}) = 1$ and $x(t) < 1$ for $t < \hat{t}$. Thus, we have $dx(\hat{t})/dt \ge 0$. But

$$\begin{aligned}\frac{dx(\hat{t})}{dt} &= -\gamma x(\hat{t}) + abm\exp(-\gamma\tau_1)(1 - x(\hat{t} - \tau_1))y(\hat{t} - \tau_1)\\ &= -\gamma + abm\exp(-\gamma\tau_1)(1 - \phi(\hat{t} - \tau_1))\psi(\hat{t} - \tau_1)\\ &< -\gamma + abm\exp(-\gamma\tau_1)\\ &< 0\end{aligned}$$

since (5) and (6). This is a contradiction. Hence, $x(t) < 1$ for $t \in [0, \tau_1]$.

Similarly we can prove $0 < y(t) < 1$ for $t \in [0, \tau_2]$. Thus, by assuming that $0 < x(t) < 1$ for $t \in [(k-1)\tau_1, k\tau_1]$ and $0 < y(t) < 1$ for $t \in [(k-1)\tau_2, k\tau_2]$ where $k > 1$ is a positive integer, (7) is obtained by induction.

Remark 2.3. Lemma 2.2 gives a sufficient condition to guarantee the positivity and boundedness of the solutions of (1) with (5). Noting (2), the condition (6) implies that $R_0 < 1$. But in fact, numerical simulations show that the solutions of (1) with (5) can be also positive and bounded even if $R_0 > 1$. Hence, with the biological background of (1) concerned, we denote

$$\Omega = \{ (a, b, c, m, \gamma, \mu, \tau_1, \tau_2) \mid \text{such that solutions of (1) with (5) are positive and bounded} \}. \quad (8)$$

MAIN RESULTS

We first give a theorem in [14] of Halanay type inequality that will be used in the sequel to derive the global asymptotical stability of the disease free equilibrium of (1) with (5).

Lemma 3.1. (Theorem 3.1 in [14]) Let A=$(a_{ij})_{n\times n}$ and $a_{ij} \geq 0$ for $i \neq j$, B=$(b_{ij})_{n\times n}$ and $b_{ij} \geq 0$, i, j = 1, 2, ..., n, −(A + B) be a nonsingular M-matrix. For $t \in (t_0, +\infty)$, let $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T$ be a solution of the following delay differential equation

$$D^+u(t) \leq Au(t) + B[u(t)]_\tau, \quad (9)$$

with initial condition $u(t) = u(s), t_0 - \tau \leq s \leq t_0$ and $u(s)$ is continuous,

where D^+ is the right hand derivative, $[u(t)]_\tau = ([u_1(t)]_\tau, [u_2(t)]_\tau, \ldots, [u_n(t)]_\tau)^T$ and $[u_i(t)]_\tau = \underset{t-\tau\leq s\leq t}{Sup}\, u_i(s)$, i = 1, 2, ..., n. Then,

$$u(t) \leq z\exp(-\lambda(t-t_0)), t \geq t_0, \quad (10)$$

provided that

$$u(s) \leq z\exp(-\lambda(s-t_0)), t_0 - \tau \leq s \leq t_0, \quad (11)$$

where $z = (z_1, z_2, \ldots, z_n)^T > 0$ and the positive number λ are determined by

$$[\lambda E + A + B\exp(\lambda\tau)]z < 0. \quad (12)$$

Remark 3.2. In view of (10), the zero solution of (9) is globally exponentially stable and consequently globally asymptotically stable if the conditions of Lemma 3.1 are satisfied.

Next we prove that the disease free equilibrium E_0 of (1) is globally asymptotically stable by Lemma 3.1.

Theorem 3.3. The disease free equilibrium E_0 is globally asymptotically stable in Ω if $R_0 < 1$.

Proof. Rewriting (1) as

$$\begin{cases} \dfrac{dx}{dt} = -\gamma x + abm\exp(-\gamma\tau_1)y(t-\tau_1) - abm\exp(-\gamma\tau_1)x(t-\tau_1)y(t-\tau_1), \\ \dfrac{dy}{dt} = -\mu y + ac\exp(-\mu\tau_2)x(t-\tau_2) - ac\exp(-\mu\tau_2)x(t-\tau_2)y(t-\tau_2). \end{cases} \quad (13)$$

From (8) we have

$$\begin{cases} \dfrac{dx}{dt} \leq -\gamma x + abm\exp(-\gamma\tau_1)[y(t)]_\tau, \\ \dfrac{dy}{dt} \leq -\mu y + ac\exp(-\mu\tau_2)[x(t)]_\tau, \end{cases} \quad (14)$$

where

$$[y(t)]_\tau = \underset{t-\tau\leq s\leq t}{Sup}\, y(s), \; [x(t)]_\tau = \underset{t-\tau\leq s\leq t}{Sup}\, x(s).$$

Comparing with (9),

$$A=\begin{pmatrix}-\gamma & 0\\ 0 & -\mu\end{pmatrix}, B=\begin{pmatrix}0 & abm\exp(-\gamma\tau_1)\\ ac\exp(-\mu\tau_2) & 0\end{pmatrix},$$

And $t_0 = 0$. Hence,

$$-(A+B)=\begin{pmatrix}\gamma & -abm\exp(-\gamma\tau_1)\\ -ac\exp(-\mu\tau_2) & \mu\end{pmatrix}.$$

We first show that −(A+B) is a nonsingular M-matrix if $R_0 < 1$. It is sufficient to prove that the successive principle minors of −(A + B) are all positive [7]. γ is positive, and

$$\det(-(A+B)) = \gamma\mu - a^2bcm\exp(-\gamma\tau_1)\exp(-\mu\tau_2).$$

Noting (2), the definition of R_0, det(−(A+B)) > 0 if $R_0 < 1$. That is, −(A+B) is a nonsingular M-matrix if $R_0 < 1$. Other conditions of A, B in Lemma 3.1 are also satisfied. Therefore, we can choose positive λ and $z = (z_1, z_2)^T > 0$ such that (12) is satisfied.

Next we show that for λ, z chosen as above, (11) can be satisfied with $t_0 = 0$ by (5). Set

$$k_1 = \sup_{s\in[-\tau,0]} \phi(s), k_2 = \sup_{s\in[-\tau,0]} \psi(s)$$

and $k_3 = k_1k_2$. Denote

$$k_4 = \frac{\max\{z_1, z_2\}}{\min\{z_1, z_2\}}.$$

Thus, we have

$$x(s) = \phi(s) \le k_3k_4z_1\exp(-\lambda s),\ y(s) = \psi(s) \le k_3k_4z_2\exp(-\lambda s)$$

for s∈ [−τ, 0]. Noting that $\bar{z} = k_3k_4z\ @(\bar{z}_1, \bar{z}_2)^T$ and λ are also solution of (12), therefore, (11) holds for $\bar{z}$ and λ.

Finally, from Lemma 3.1 we have

$$x(t) \le \bar{z}_1\exp(-\lambda t),\ y(t) \le \bar{z}_2\exp(-\lambda t)$$

for $t \ge 0$. That is, by comparison arguments, the zero solution of (13), which is the disease free equilibrium of (1), is globally asymptotically stable in Ω when $R_0 < 1$.

Remark 3.4. The exponential convergence rate of solutions of (1) with (5) convergent to the disease free equilibrium is at least λ if $R_0 < 1$ from the proof of Theorem 3.3. It can be used to estimate the time before the disease ceases its prevalence in human population if $R_0 < 1$.

Remark 3.5. The global asymptotical stability of the disease free equilibrium of (1) has been analyzed in [13] by constructing Liapunov functions. The proof of this paper is more simple and direct.

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REFERENCES

1. Baker CTH; Development and application of Halanay-type theory: Evolutionary differential and difference equations with time lag. J. Comput. Appl. Math., 2010; 234: 2663-2682.
2. Chen Y, Tian R; Exponential stability of impulsive discrete systems with multiple delays. J. Networks, 2013; 8: 2564-2571.
3. Gopalsamy K; Stability and Oscillations in Delay Differential Equations of Population Dynamics. Springer, 1993.
4. Gu K, Kharitonov VL, Chen J; Stability of Time Delay Systems. Springer, 2003.
5. Hethcote HW; The mathematics of infectious diseases. SIAM Reviews, 2000; 42: 599-653.
6. Kuang Y; Delay Differential equations with Applications in Population Biology. Academic Press, Boston, 1993.
7. Plemmons RJ; M-matrix characterizations I: Nonsingular M-matrix, Linear Algebra and its Applications. 1977; 18: 175-188.
8. Qi L, Cui J; The stability of an SEIRS model with nonlinear incidence rate, vertical transmission and time delay. Appl. Math. Comput., 2013; 221: 360-366.
9. Ruan SG, Xiao D, Beier JC; On the delayed Ross-Macdonald model for malaria transmission. Bull. Math. Biol.,

2008; 70: 1098-1114.
10. Sariyasa; Application of Halanay inequality to the establishment of the exponential stability of delayed compartmental model. J. Indonesia Math. Soc., 2007; 13: 191-196.
11. Wan H, Cui J; A malaria model with two delays. Discrete Dynamics in Nature and Society, 2013; Paper ID 601265, 8 pages.
12. Wei J, Cui J; Dynamics of SIS epidemic model with the standard incidence rate and saturated treatment function. Int. J. Biomath., 2012; 5: 12603, 18 pages.
13. Wu C, Jiang Z; Global stability of disease free equilibrium of a delayed model for malaria transmission. Int. J. Math. Anal., 2012; 6: 1877-1881.
14. Xu D, Yang Z; Impulsive delay differential inequality and stability of neural networks. J. Math. Anal. Appl., 2005; 305: 107-120.
15. Zhou X, Cui J, Zhang J; Global results for a cholera model with imperfect vaccination. J. Franklin Institute, 2012; 349: 770-791.