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Classification of the Solutions to the third-order homogeneous linear difference equation with constant coefficients

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Abstract: This paper mainly studies classification of the solutions to the third-order homogeneous linear difference equation with constant coefficients. First, we determine the generating function of it. Then, we use the complete discrimination system for polynomial to get classification of the solutions for the considered equation.
Keywords: Generating function, Difference equation, Complete discrimination system for polynomial.

INTRODUCTION

Generating function plays an important role in many fields ,such as probability theory, combinatorial mathematics, programming and algorithm design and digital signal processing[1,2]. Generating function method is also a powerful tool for studying difference equations.

Complete discrimination system for polynomial can be applied to solve all kinds of the partial differential equations [3,4,5,6]. In this paper, we use this method to solve the difference equation. It is an initiative and there is no one else that does so.
In this paper, we first determine the generating function of the unknown function in the fourth-order homogeneous linear difference equation with constant coefficients.

Then, we use the complete discrimination system for polynomial to get classification of the solutions for the considered equation.

GENERATING FUNCTION OF THE THIRD-ORDER HOMOGENEOUS LINEAR DIFFERENCE EQUATION

In this paper, we take into account the third-order homogeneous linear difference equation, and it reads as

$$a_n + pa_{n-1} + qa_{n-2} + \omega a_{n-3} = 0$$

The characteristic equation of the above formula is

$$r^3 + pr^2 + qr + \omega = 0$$

Where p , q , ω are real numbers and $\omega \neq 0$.

Let $\{a_n\}$ and $\{b_n\} = \{1, p, q, \omega, \cdots\}$, then we do convolution calculation to them[2].
We get
$c_n = \{a_0, a_0 p + a_1, a_0 q + a_1 p + a_2, a_0\omega + a_1 q + a_2 p + a_3, \cdots\} = \{a_0, a_0 p + a_1, a_0 q + a_1 p + a_2, 0, \cdots\}$ Let
$\sum_{n=0}^{\infty} c_n x^n = (\sum_{n=0}^{\infty} a_n x^n)(\sum_{n=0}^{\infty} b_n x^n)$, so we obtain the generating function of $\{a_n\}$ as follows

$$\sum_{n=0}^{\infty} a_n x^n = \frac{\sum_{n=0}^{\infty} c_n x^n}{\sum_{n=0}^{\infty} b_n x^n} = \frac{a_0 + (a_0 p + a_1)x + (a_0 q + a_1 p + a_2)x^2}{1 + px + qx^2 + \omega x^3} \quad (1)$$

CLASSIFICATION OF THE SOLUTIONS TO THE CONSIDERED EQUATION

According to complete discrimination system for polynomial, there are nine cases as follows. Let

$$\Delta = -\frac{27}{\omega^6}\left(\frac{2}{27}q^3 + \omega^2 - \frac{p\omega}{3}\right)^2 - \frac{4}{\omega^4}\left(p\omega - \frac{1}{3}q^2\right)^2$$

$$D_1 = \frac{1}{\omega^2}\left(p\omega - \frac{1}{3}q^2\right), \quad F(x) = 1 + px + qx^2 + \omega x^3$$

(1). If $\Delta > 0$, $D_1 < 0$, the characteristic equation has three different real roots $\gamma_1, \gamma_2, \gamma_3$. That is

$$F(x) = (1-\gamma_1 x)(1-\gamma_2 x)(1-\gamma_3 x)$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty} a_n x^n &= \frac{C_1}{1-\gamma_1 x} + \frac{C_2}{1-\gamma_2 x} + \frac{C_3}{1-\gamma_3 x} \\ &= C_1\sum_{n=0}^{\infty}(\gamma_1 x)^n + C_2\sum_{n=0}^{\infty}(\gamma_2 x)^n + C_3\sum_{n=0}^{\infty}(\gamma_3 x)^n \\ &= \sum_{n=0}^{\infty}(C_1\gamma_1^n + C_2\gamma_2^n + C_3\gamma_3^n)x^n\end{aligned}$$

Therefore, we get

$$a_n = C_1\gamma_1^n + C_2\gamma_2^n + C_3\gamma_3^n \tag{2}$$

(2). If $\Delta = 0$, $D_1 < 0$, the characteristic equation has a simple real roots γ_1, and a double real root γ_2. That is

$$F(x) = (1-\gamma_1 x)(1-\gamma_2 x)^2$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty} a_n x^n &= \frac{C_1}{1-\gamma_1 x} + \frac{C_2}{1-\gamma_2 x} + \frac{C_3}{(1-\gamma_2 x)^2} \\ &= C_1\sum_{n=0}^{\infty}(\gamma_1 x)^n + C_2\sum_{n=0}^{\infty}(\gamma_2 x)^n + C_3\sum_{n=0}^{\infty}(n+1)\gamma_2^n x^n \\ &= \sum_{n=0}^{\infty}[C_1\gamma_1^n + C_2\gamma_2^n + C_3(n+1)\gamma_2^n]x^n\end{aligned}$$

Therefore, we get

$$a_n = C_1\gamma_1^n + [C_2 + C_3(n+1)]\gamma_2^n \tag{3}$$

(3) If $\Delta = 0$, $D_1 = 0$, the characteristic equation has a triple real root γ_1.
That is

$$F(x) = (1-\gamma_1 x)^3$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty} a_n x^n &= \frac{C_1}{1-\gamma_1 x} + \frac{C_2}{(1-\gamma_1 x)^2} + \frac{C_3}{(1-\gamma_3 x)^2} \\ &= C_1\sum_{n=0}^{\infty}(\gamma_1 x)^n + C_2\sum_{n=0}^{\infty}(n+1)\gamma_1^n x^n + C_3\sum_{n=0}^{\infty}\frac{1}{2}(n+1)(n+2)\gamma_1^n x^n \\ &= \sum_{n=0}^{\infty}[C_1\gamma_1^n + C_2(n+1)\gamma_1^n + C_3\frac{1}{2}(n+1)(n+2)\gamma_1^n]x^n\end{aligned}$$

Therefore, we get

$$a_n = [C_1 + C_2(n+1) + \frac{1}{2}C_3(n+1)(n+2)]\gamma_1^n \tag{4}$$

(4) If $\Delta < 0$, the characteristic equation has a simple real root γ_1 and a pair of conjugate imaginary roots γ_2, γ_3. That is

$$F(x) = (1-\gamma_1 x)(1-\gamma_2 x)(1-\gamma_3 x)$$

Then

$$\sum_{n=0}^{\infty} a_n x^n = \frac{C_1}{1-\gamma_1 x} + \frac{C_2}{1-\gamma_2 x} + \frac{C_3}{1-\gamma_3 x}$$
$$= C_1 \sum_{n=0}^{\infty} (\gamma_1 x)^n + C_2 \sum_{n=0}^{\infty} (\gamma_2 x)^n + C_3 \sum_{n=0}^{\infty} (\gamma_3 x)^n$$
$$= \sum_{n=0}^{\infty} (C_1\gamma_1^n + C_2\gamma_2^n + C_3\gamma_3^n) x^n$$

Therefore, we get

$$a_n = C_1\gamma_1^n + C_2\gamma_2^n + C_3\gamma_3^n = C_1\gamma_1^n + C_2\gamma_2^n + \bar{C}_2\bar{\gamma}_2^n$$
$$= C_1\gamma_1^n + 2\,\mathrm{Re}[|C_2|(\cos\theta_0 + i\sin\theta_0)|\gamma_2|^n(\cos n\varphi + i\sin n\varphi)]$$
$$= C_1\gamma_1^n + 2|\gamma_2|^n |C_2|(\cos\theta_0\cos n\varphi - \sin\theta_0 \sin n\varphi) = C_1\gamma_1^n + |\gamma_2|^n (C_2'\cos n\varphi + C_3'\sin n\varphi) \tag{5}$$

where $C_2 = |C_2|(\cos\theta_0 + i\sin\theta_0)$, $\gamma_2 = |\gamma_2|(\cos\varphi + i\sin\varphi)$, $C_2 = \bar{C}_3$.

CONCLUSIONS

In this paper, we determine the generating function of the unknown function in the third-order homogeneous linear difference equation with constant coefficients. Then, we use the complete discrimination system for polynomial to get classification of the solutions and obtain the solutions (2) (3) (4) (5) for the considered equation. This paper has introduced a simple and useful method to solve the homogeneous linear difference equations with constant coefficients.

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